

# DUALITY THEOREMS FOR COINVARIANT SUBSPACES OF $H^1$

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ABSTRACT. Let  $\theta$  be an inner function satisfying the connected level set condition of B. Cohn, and let  $K_\theta^1$  be the shift-coinvariant subspace of the Hardy space  $H^1$  generated by  $\theta$ . We describe the dual space to  $K_\theta^1$  in terms of a bounded mean oscillation with respect to the Clark measure  $\sigma_\alpha$  of  $\theta$ . Namely, we prove that  $(K_\theta^1 \cap zH^1)^* = \text{BMO}(\sigma_\alpha)$ . The result implies a two-sided estimate for the operator norm of a finite Hankel matrix of size  $n \times n$  via  $\text{BMO}(\mu_{2n})$ -norm of its standard symbol, where  $\mu_{2n}$  is the Haar measure on the group  $\{\xi \in \mathbb{C} : \xi^{2n} = 1\}$ .

## 1. INTRODUCTION

A bounded analytic function  $\theta$  in the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  is called inner if  $|\theta(z)| = 1$  for almost all points  $z$  on the unit circle  $\mathbb{T}$  in the sense of angular boundary values. With every inner function  $\theta$  we associate the shift-coinvariant [19] subspace  $K_\theta^p$  of the Hardy space  $H^p$ ,

$$K_\theta^p = H^p \cap \bar{z}\theta\overline{H^p}, \quad 1 \leq p \leq \infty. \quad (1)$$

As usual, functions in  $H^p$  are identified with their angular boundary values on the unit circle  $\mathbb{T}$ ; formula (1) means that  $f \in K_\theta^p$  if  $f \in H^p$  and there is  $g \in H^p$  such that  $f(z) = \bar{z}\theta(z)\overline{g(z)}$  for almost all points  $z \in \mathbb{T}$ . An inner function  $\theta$  is said to be *one-component* if its sublevel set  $\Omega_\delta = \{z \in \mathbb{D} : |\theta(z)| < \delta\}$  is connected for a positive number  $\delta < 1$ . This class of inner functions was introduced by B.Cohn [12] in 1982. It is very useful in studying Carleson-type embeddings  $K_\theta^p \hookrightarrow L^p(\mu)$  and Riesz bases of reproducing kernels in  $K_\theta^p$ , see [3–5, 7, 12, 13, 17, 26] for results and further references.

In this paper we describe the dual space to the space  $K_\theta^1$  generated by a one-component inner function  $\theta$ . Our main result is the following formula:

$$(K_\theta^1 \cap zH^1)^* = \text{BMO}(\sigma_\alpha), \quad (2)$$

where  $\sigma_\alpha$  denotes the Clark measure of the inner function  $\theta$ . Below we state this result formally and apply it to the boundedness problem for truncated Hankel operators.

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**1.1. Clark measures of one-component inner functions.** Let  $\theta$  be a non-constant inner function in the open unit disk  $\mathbb{D}$ . For each complex number  $\alpha$  of unit modulus the function  $\operatorname{Re}\left(\frac{\alpha+\theta}{\alpha-\theta}\right)$  is positive and harmonic in  $\mathbb{D}$ . Hence there exists the unique positive Borel measure  $\sigma_\alpha$  supported on the unit circle  $\mathbb{T}$  such that

$$\operatorname{Re} \frac{\alpha + \theta(z)}{\alpha - \theta(z)} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\xi}z|^2} d\sigma_\alpha(\xi), \quad z \in \mathbb{D}. \quad (3)$$

The measures  $\{\sigma_\alpha\}_{|\alpha|=1}$  are usually referred to as Clark measures of the inner function  $\theta$  due to seminal work [11] of D. N. Clark where their close connection to rank-one perturbations of singular unitary operators was discovered. For a modern exposition of this topic and subsequent results see survey [21].

Each Clark measure  $\sigma_\alpha$  of an inner function  $\theta$  is singular with respect to the Lebesgue measure on the unit circle  $\mathbb{T}$ . Conversely if  $\mu$  is a finite positive Borel singular measure supported on  $\mathbb{T}$  and  $|\alpha| = 1$ , then there exists the unique inner function  $\theta$  satisfying (3) with  $\sigma_\alpha = \mu$ . Thus, there is one-to-one correspondence between inner functions in the unit disk  $\mathbb{D}$  and singular measures on the unit circle  $\mathbb{T}$ . It was unknown which singular measures on  $\mathbb{T}$  correspond to the Clark measures of one-component inner functions. We fill this gap in Theorem 1 below.

For every Borel measure  $\mu$  on the unit circle  $\mathbb{T}$  denote by  $a(\mu)$  the set of isolated atoms of  $\mu$ . Then the set  $\rho(\mu) = \operatorname{supp} \mu \setminus a(\mu)$  consists of accumulating points in the support  $\operatorname{supp} \mu$  of  $\mu$ . We will say that an atom  $\xi \in a(\mu)$  has two neighbours in  $a(\mu)$  if there is an open arc  $(\xi_-, \xi_+)$  of the unit circle  $\mathbb{T}$  with endpoints  $\xi_\pm \in a(\mu)$  such that  $\xi$  is the only point in  $(\xi_-, \xi_+) \cap \operatorname{supp} \mu$ . By  $m$  we will denote the Lebesgue measure on  $\mathbb{T}$  normalized so that  $m(\mathbb{T}) = 1$ .

**Theorem 1.** *Let  $|\alpha| = 1$ . The following conditions are necessary and sufficient for a Borel measure  $\mu$  to be the Clark measure  $\sigma_\alpha$  of a one-component inner function:*

- (a)  $\mu$  is a discrete measure on  $\mathbb{T}$  with isolated atoms,  $m(\operatorname{supp} \mu) = 0$ , every atom  $\xi \in a(\mu)$  has two neighbours  $\xi_\pm$  in  $a(\mu)$ , and every connected component of  $\mathbb{T} \setminus \rho(\mu)$  contains atoms of  $\mu$ ;
- (b)  $A_\mu |\xi - \xi_\pm| \leq \mu\{\xi\} \leq B_\mu |\xi - \xi_\pm|$  for all  $\xi \in a(\mu)$  and some  $A_\mu > 0$ ,  $B_\mu < \infty$ ;
- (c) the discrete Hilbert transform  $(H_\mu 1)(z) = \int_{\mathbb{T} \setminus \{z\}} \frac{d\mu(\xi)}{1 - \bar{\xi}z}$  is bounded on  $a(\mu)$ : we have  $|(H_\mu 1)(z)| \leq C_\mu$  for all  $z \in a(\mu)$ .

The necessity of conditions (a) and (b) in Theorem 1 is well-known. I would like to thank A. D. Baranov who tell me the fact that condition (c) is necessary as well. The proof of sufficiency part in Theorem 1 relies on a characterization of one-component inner functions in terms of their derivatives which is due to A. B. Aleksandrov [3].

**1.2. The main result.** Having a description of the Clark measures of one-component inner functions, we now turn back to formula (2). For a measure  $\mu$  with properties (a) – (c) define the space  $\operatorname{BMO}(\mu)$  by

$$\operatorname{BMO}(\mu) = \left\{ b \in L^1(\mu) : \|b\|_{\mu^*} = \sup_{\Delta} \frac{1}{\mu(\Delta)} \int_{\Delta} |b - \langle b \rangle_{\Delta, \mu}| d\mu < \infty \right\},$$

where  $\Delta$  runs over all arcs of  $\mathbb{T}$  with non-zero mass  $\mu(\Delta)$  and  $\langle b \rangle_{\Delta, \mu} = \frac{1}{\mu(\Delta)} \int_{\Delta} b d\mu$  is the standard integral mean of  $b$  on  $\Delta$ . The following theorem is the main result of the paper.

**Theorem 2.** *Let  $\theta$  be a one-component inner function and let  $\sigma_\alpha$  be its Clark measure. We have  $(K_\theta^1 \cap zH^1)^* = \text{BMO}(\sigma_\alpha)$ . That is, for every continuous linear functional  $\Phi$  on  $K_\theta^1 \cap zH^1$  there exists a function  $b \in \text{BMO}(\sigma_\alpha)$  such that  $\Phi = \Phi_b$ , where*

$$\Phi_b : F \mapsto \int_{\mathbb{T}} F b d\sigma_\alpha, \quad F \in K_\theta^1 \cap zH^\infty. \quad (4)$$

*Conversely, for every function  $b \in \text{BMO}(\sigma_\alpha)$  the functional  $\Phi_b$  is the densely defined continuous linear functional on  $K_\theta^1 \cap zH^1$  with norm comparable to  $\|b\|_{\sigma_\alpha^*}$ .*

Every measure  $\mu$  with properties (a), (b) from Theorem 1 generates the doubling metric space  $(\text{supp } \mu, |\cdot|, \mu)$  in the sense of R. Coifman and G. Weiss [14]. For such measures  $\mu$  we have  $H_{at}^1(\mu)^* = \text{BMO}(\mu)$ , where  $H_{at}^1(\mu)$  is the corresponding atomic Hardy space,

$$H_{at}^1(\mu) = \left\{ \sum_k \lambda_k a_k : a_k \text{ are } \mu\text{-atoms, } \sum_k |\lambda_k| < \infty \right\}. \quad (5)$$

By a  $\mu$ -atom we mean a complex-valued function  $a \in L^\infty(\mu)$  supported on an arc  $\Delta$  of  $\mathbb{T}$ , with  $\|a\|_{L^\infty(\mu)} \leq 1/\mu(\Delta)$ , and such that  $\langle a \rangle_{\Delta, \mu} = 0$ . The norm of  $f \in H_{at}^1(\mu)$  is the infimum of  $\sum_k |\lambda_k|$  over all possible representations  $f = \sum_k \lambda_k a_k$  of  $f$  as a sum of  $\mu$ -atoms. We see from Theorem 1 that Theorem 2 admits the following equivalent reformulation.

**Theorem 2'.** *Let  $\mu$  be a measure with properties (a) – (c). Then  $f \in H_{at}^1(\mu)$  if and only if  $f$  admits the analytic continuation to the unit open disk  $\mathbb{D}$  as a function  $F \in K_\theta^1 \cap zH^1$ , where  $\theta$  is the inner function with the Clark measure  $\sigma_\alpha = \mu$ . Moreover, such a function  $F$  is unique and the norms  $\|f\|_{H_{at}^1(\mu)}$ ,  $\|F\|_{L^1(\mathbb{T})}$  are comparable.*

For the counting measure  $\mu = \delta_{\mathbb{Z}}$  on the set of integers  $\mathbb{Z}$  Theorem 2' follows from the results by C. Eoff [15], S. Boza and M. Carro [8]. They proved that  $f \in H_{at}^1(\mathbb{Z})$  if and only if  $f$  admits the analytic continuation to the complex plane  $\mathbb{C}$  as a function from the Paley-Wiener space  $PW_{[0, 2\pi]}^1$ . It seems difficult to adapt the technique of [8] (where convolution operators were used to relate  $H_{at}^1(\mathbb{Z})$  and  $\text{Re } H^1(\mathbb{R})$ ) for the general measures  $\mu$  with properties (a) – (c). Instead we give a complex-analytic proof based on the Cauchy-type formula

$$\int_{\Delta} F(\xi) d\sigma_\alpha(\xi) = \oint_{\Gamma} \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} dz, \quad (6)$$

where  $\Delta$  is an arc of  $\mathbb{T}$ ,  $\Gamma$  is a simple closed contour in  $\mathbb{C}$  which intersects  $\mathbb{T}$  at the endpoints of  $\Delta$ , and  $F \in K_\theta^1 \cap zH^1$ . Once we have a good estimate for the function  $\frac{F(z)/z}{1 - \bar{\alpha}\theta(z)}$  on  $\Gamma$ , formula (6) gives us an upper bound for the mean  $\langle F \rangle_{\Delta, \sigma_\alpha}$  on the arc  $\Delta$ . Then we can use a standard Calderón-Zigmund decomposition to obtain the representation of  $F$  as a sum of atoms with respect to the measure  $\sigma_\alpha$ . The idea of using a contour integration is taken from the classical proof of atomic decomposition of  $\text{Re}(zH^1)$ , where the contour  $\Gamma$  comes from the Lusin-Privalov construction. In our situation we have to modify this construction so that the contour  $\Gamma$  does not approach the subsets of the unit disk  $\mathbb{D}$  where the function  $|\alpha - \theta|$  is small.

**1.3. Truncated Hankel operators.** One of important applications of the classical Fefferman duality theorem is the boundedness criterium for Hankel operators on the Hardy space  $H^2$ . Theorem 1 yields a similar criterium for truncations of Hankel operators to coinvariant subspaces of  $H^2$ .

Let  $\theta$  be an inner function and let  $K_\theta^2$  be the corresponding coinvariant subspace (1) of the Hardy space  $H^2$ . Denote by  $P_\theta$  the orthogonal projection in  $L^2(\mathbb{T})$  to the subspace  $\overline{zK_\theta^2} = \{f \in L^2(\mathbb{T}) : f = \overline{zg}, g \in K_\theta^2\}$ . The truncated Hankel operator with symbol  $\varphi \in L^2(\mathbb{T})$  is the densely defined operator  $\Gamma_\varphi : K_\theta^2 \rightarrow \overline{zK_\theta^2}$ ,

$$\Gamma_\varphi : f \mapsto P_\theta(\varphi f), \quad f \in K_\theta^2. \quad (7)$$

The symbol  $\varphi$  of  $\Gamma_\varphi$  is not unique. However, it is easy to check that every truncated Hankel operator on  $K_\theta^2$  has the unique “standard” symbol  $\varphi \in \overline{K_{\theta^2}^2} \cap zH^2$ , which plays the same role as the antianalytic symbol of a Hankel operator on  $H^2$ .

Two special cases of truncated Hankel operators are of traditional interest in the operator theory. If  $\theta = z^n$ , then the operators defined by (7) are classical Hankel matrices of size  $n \times n$ . Indeed, in this situation the space  $K_\theta^2$  consists of analytic polynomials of degree at most  $n - 1$  and the entries of the matrix of  $\Gamma_\varphi$  in the standard bases of  $K_\theta^2$  and  $\overline{zK_\theta^2}$  depend only on the difference  $k - l$ : we have  $(\Gamma_\varphi z^k, \overline{z}^{l+1}) = \hat{\varphi}(-k - l - 1)$  for  $0 \leq k, l \leq n - 1$ . Similarly, for the inner function  $\theta_a : z \mapsto e^{iaz}$  in the upper half-plane  $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$  the corresponding coinvariant subspace  $K_{\theta_a}^2$  of the Hardy space  $H^2(\mathbb{C}_+)$  can be identified with the Paley-Wiener space  $\text{PW}_{[0,a]}^2$ ; truncated Hankel operators on  $\text{PW}_{[0,a]}^2$  are unitarily equivalent to the Wiener-Hopf convolution operators on the interval  $[0, a]$ , see [9, 23].

The question for which symbols  $\varphi \in L^2(\mathbb{T})$  the truncated Hankel operator  $\Gamma_\varphi$  is bounded on  $K_\theta^2$  (and how to estimate its operator norm in terms of  $\varphi$ ) admits several equivalent reformulations. It has been studied in [5, 6, 9, 18, 23, 24], see the discussion in Section 4. Most of known results are Nehary-type theorems: under certain restrictions they affirm the existence of a bounded symbol for a bounded truncated Hankel/Toeplitz operator with control of the norms. Until now, the only BMO-type criterium for truncated Hankel operators was known. In 2011, M. Carlsson [9] proved that a Hankel operator  $\Gamma_\varphi$  on  $\text{PW}_{[0,\pi]}^2$  with standard symbol  $\varphi$  is bounded if and only if the sequence  $\{\varphi(n)\}_{n \in \mathbb{Z}}$  lies in the space  $\text{BMO}(\mathbb{Z})$ . Recall that we have  $\text{PW}_{[0,\pi]}^2 = K_{\theta_\pi}^2$  for the special one-component inner function  $\theta_\pi : z \mapsto e^{i\pi z}$  in the upper half-plane  $\mathbb{C}_+$ . The counting measure  $\delta_{\mathbb{Z}}$  on  $\mathbb{Z}$  can be regarded as the Clark measure  $\nu_1$  for the inner function  $\theta_\pi^2$  (for every inner function  $\theta$  the Clark measures of  $\theta^2$  will be denoted by  $\nu_\alpha$ ; from (3) we see that  $\nu_\alpha = (\sigma_\alpha + \sigma_{-\bar{\alpha}})/2$ ,  $|\alpha| = 1$ ). Therefore the following result is a generalization of the criterium by M. Carlsson.

**Theorem 3.** *Let  $\theta$  be a one-component inner function, and let  $\nu_\alpha$  be the Clark measure of the inner function  $\theta^2$ . The truncated Hankel operator  $\Gamma_\varphi : K_\theta^2 \rightarrow \overline{zK_\theta^2}$  with standard symbol  $\varphi$  is bounded if and only if  $\varphi \in \text{BMO}(\nu_\alpha)$ . Moreover, we have*

$$c_1 \|\varphi\|_{\nu_\alpha^*} \leq \|\Gamma_\varphi\| \leq c_2 \|\varphi\|_{\nu_\alpha^*}, \quad (8)$$

for some constants  $c_1, c_2$  depending only on the inner function  $\theta$ .

Similarly, one can describe compact truncated Hankel operators in terms of their standard symbols: we have  $\Gamma_\varphi \in S_\infty$  if and only if  $\varphi \in \text{VMO}(\nu_\alpha)$ , see Section 4.

Theorem 3 for the inner function  $\theta = z^n$  yields the following interesting corollary for finite Hankel matrices.

**Corollary 1.** *Let  $\Gamma = (\gamma_{j+k})_{0 \leq k, j \leq n-1}$  be a Hankel matrix of size  $n \times n$ ; consider its standard symbol  $\varphi = \gamma_0 \bar{z} + \gamma_1 \bar{z}^2 + \dots \gamma_{2n-2} \bar{z}^{2n-1}$ . We have*

$$c_1 \|\varphi\|_{\mu_{2n}^*} \leq \|\Gamma\| \leq c_2 \|\varphi\|_{\mu_{2n}^*}, \quad (9)$$

where the constants  $c_1, c_2$  do not depend on  $n$  and  $\mu_{2n} = \frac{1}{2n} \sum \delta_{z\sqrt[n]{1}}$  is the Haar measure on the group  $\{\xi \in \mathbb{C} : \xi^{2n} = 1\}$ .

Corollary 1 implies the boundedness criterium for the standard Hankel operators on  $H^2$ . Recall that the Hankel operator  $H_\varphi : H^2 \rightarrow \overline{zH^2}$  with symbol  $\varphi \in L^2(\mathbb{T})$  is densely defined by

$$H_\varphi : f \mapsto P_-(\varphi f), \quad f \in H^\infty,$$

where  $P_-$  denotes the orthogonal projection in  $L^2(\mathbb{T})$  to  $\overline{zH^2}$ . It follows from the classical Fefferman duality theorem that  $H_\varphi$  is bounded if and only if its antianalytic symbol  $P_- \varphi$  lies in  $\text{BMO}(\mathbb{T})$ . Moreover, the operator norm of  $H_\varphi$  is comparable to  $\|P_- \varphi\|_*$ , the norm of  $P_- \varphi$  in  $\text{BMO}(\mathbb{T})$ . Taking the limit in (9) as  $n \rightarrow \infty$  one can prove the estimate  $c_1 \|\varphi\|_* \leq \|H_\varphi\| \leq c_2 \|\varphi\|_*$  for every antianalytic polynomial  $\varphi$ . This is already sufficient to obtain the general version of the boundedness criterium for Hankel operators on  $H^2$ , see details in Section 4.

## 2. PROOF OF THEOREM 1

**2.1. Preliminaries.** Given an inner function  $\theta$ , denote by  $\rho(\theta)$  its boundary spectrum, that is, the set of points  $\zeta \in \mathbb{T}$  such that  $\liminf_{z \rightarrow \zeta, z \in \mathbb{D}} |\theta(z)| = 0$ . In this paper we always assume that  $\rho(\theta) \neq \mathbb{T}$ , because this is so for one-component inner functions and for functions satisfying condition (a) in Theorem 1 (see Lemma 2.1 below). As is well-known, the function  $\theta$  admits the analytic continuation from the open unit disk  $\mathbb{D}$  to the open domain  $\mathbb{D} \cup G_\theta$ , where  $G_\theta = (\mathbb{T} \setminus \rho(\theta)) \cup \{z : |z| > 1, \theta(1/\bar{z}) \neq 0\}$ . The analytic continuation is given by

$$\theta(z) = \frac{1}{\overline{\theta(1/\bar{z})}}, \quad z \in G_\theta. \quad (10)$$

Moreover,  $\mathbb{D} \cup G_\theta$  is the maximal domain to which  $\theta$  can be extended analytically. We need the following known lemma.

**Lemma 2.1.** *Let  $\theta$  be an inner function with the Clark measure  $\sigma_\alpha$ ,  $|\alpha| = 1$ . Then  $\rho(\theta) = \rho(\sigma_\alpha)$ . A point  $z \in \mathbb{T} \setminus \rho(\theta)$  belongs to  $\text{supp } \sigma_\alpha$  if and only if  $\theta(z) = \alpha$ . Moreover, in the latter case we have  $z \in a(\sigma_\alpha)$  and  $\sigma_\alpha\{z\} = |\theta'(z)|^{-1}$ .*

**Proof.** As is easy to see from formula (3), we have

$$\frac{\alpha + \theta(z)}{\alpha - \theta(z)} = \int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \xi z} d\sigma_\alpha(\xi) + i \operatorname{Im} \frac{\alpha + \theta(0)}{\alpha - \theta(0)}, \quad z \in \mathbb{D} \cup G_\theta. \quad (11)$$

Since  $\theta$  is analytic on  $\mathbb{D} \cup G_\theta$ , a point  $z \in \mathbb{T} \setminus \rho(\theta)$  belongs to  $\text{supp } \sigma_\alpha$  if and only if  $\theta(z) = \alpha$ , and in the latter case there is no other points of  $\text{supp } \sigma_\alpha$  in a small neighbourhood of  $z$ . Hence  $z \in a(\sigma_\alpha)$  and we see from (11) that

$$\sigma_\alpha\{z\} = (\bar{\alpha}z\theta'(z))^{-1} = |\theta'(z)|^{-1}.$$

It follows that  $\mathbb{T} \setminus \rho(\theta) \subset \mathbb{T} \setminus \rho(\sigma_\alpha)$ . For every  $z \in \mathbb{T} \setminus \rho(\sigma_\alpha)$  either  $z$  is an isolated atom of  $\sigma_\alpha$  or  $z \notin \text{supp } \sigma_\alpha$ . In both cases formula (11) shows that the function

$\theta$  admits the analytic continuation from  $\mathbb{D}$  to a small neighbourhood of  $z$ . Hence  $z \in \mathbb{T} \setminus \rho(\theta)$  and we have  $\rho(\theta) = \rho(\sigma_\alpha)$ .  $\square$

The following result is in [3], see Theorem 1.11 and Remark 2 after its proof.

**Theorem** (A. B. Aleksandrov). *An inner function  $\theta$  is one-component if and only if it satisfies the following conditions:*

- (A1)  $m(\rho(\theta)) = 0$  and  $|\theta'|$  is unbounded on every open arc  $\Delta \subset \mathbb{T} \setminus \rho(\theta)$  such that  $\overline{\Delta} \cap \rho(\theta) \neq \emptyset$ ;
- (A2)  $\theta$  satisfies the estimate  $|\theta''(\xi)| \leq C|\theta'(\xi)|^2$  for all  $\xi \in \mathbb{T} \setminus \rho(\theta)$ .

**2.2. Proof of Theorem 1.** Essentially, we will show that conditions (a) – (c) in Theorem 1 are equivalent to conditions (A1), (A2) above.

**Necessity.** Let  $\theta$  be a one-component inner function and let  $\sigma_\alpha$  be its Clark measure. By Lemma 2.1 we have  $\rho(\theta) = \rho(\sigma_\alpha)$ . It was proved in [3] that  $m(\rho(\theta)) = 0$  and  $\sigma_\alpha(\rho(\theta)) = 0$ . Hence  $\sigma_\alpha$  is a discrete measure with isolated atoms and we have  $m(\text{supp } \sigma_\alpha) = 0$ . Let  $\Delta$  be a connected component of the set  $\mathbb{T} \setminus \rho(\sigma_\alpha) = \mathbb{T} \setminus \rho(\theta)$ . By property (A1) the argument of  $\theta$  on  $\Delta$  is a monotonic function unbounded near both endpoints of  $\Delta$ . It follows that the arc  $\Delta$  contains infinitely many points  $\xi_k$  such that  $\theta(\xi_k) = \alpha$ . Enumerate these points clockwise by integer numbers. We see from Lemma 2.1 that  $\xi_k \in a(\sigma_\alpha)$  for all  $k \in \mathbb{Z}$  and every atom  $\xi_k$  has two neighbours  $\xi_{k-1}, \xi_{k+1}$ . This shows that the measure  $\sigma_\alpha$  satisfies condition (a). The fact that  $\sigma_\alpha$  satisfies condition (b) follows from Lemma 5.1 of [7]. Now check condition (c). Fix an atom  $\xi_0 \in a(\sigma_\alpha)$ . From (11) we see that

$$\frac{1}{1 - \overline{\alpha}\theta(z)} = \int_{\mathbb{T}} \frac{d\sigma_\alpha(\xi)}{1 - \xi z} + c_\alpha, \quad z \in \mathbb{D} \cup G_\theta, \quad (12)$$

where  $c_\alpha = \overline{\alpha\theta(0)}/(1 - \overline{\alpha\theta(0)})$ . Hence,

$$(H_{\sigma_\alpha} 1)(\xi_0) + c_\alpha = \lim_{z \rightarrow \xi_0} \left( \frac{1}{1 - \overline{\alpha}\theta(z)} - \frac{\sigma_\alpha\{\xi_0\}}{1 - \xi_0 z} \right).$$

Consider the analytic function  $k_{\xi_0} : z \mapsto \frac{1 - \overline{\alpha}\theta(z)}{1 - \xi_0 z}$  on the domain  $\mathbb{D} \cup G_\theta$ . We have

$$\begin{aligned} (H_{\sigma_\alpha} 1)(\xi_0) + c_\alpha &= \lim_{z \rightarrow \xi_0} \frac{1}{1 - \xi_0 z} \left( \frac{1}{k_{\xi_0}(z)} - \frac{1}{k_{\xi_0}(\xi_0)} \right) \\ &= -\frac{\xi_0 k'_{\xi_0}(\xi_0)}{k_{\xi_0}^2(\xi_0)} = -\frac{\alpha\theta''(\xi_0)}{2\theta'(\xi_0)^2}. \end{aligned} \quad (13)$$

From here and the estimate in (A2) we see that  $H_{\sigma_\alpha} 1$  is bounded on  $a(\sigma_\alpha)$ . Surprisingly simple relation (13) between the discrete Hilbert transform  $H_{\sigma_\alpha} 1$  and the inner function  $\theta$  is the key observation in the proof.

**Sufficiency.** Let  $\mu$  be a measure with properties (a) – (c). Construct the inner function  $\theta$  with the Clark measure  $\sigma_\alpha = \mu$ . To prove that  $\theta$  is a one-component inner function we will check conditions (A1) and (A2).

By Lemma 2.1 we have  $\rho(\theta) = \rho(\sigma_\alpha)$ . Hence  $m(\rho(\theta)) = 0$  by property (a) of the measure  $\sigma_\alpha$ . Let  $\Delta$  be an open arc of  $\mathbb{T}$  such that  $\Delta \subset \mathbb{T} \setminus \rho(\theta)$  and  $\overline{\Delta} \cap \rho(\theta) \neq \emptyset$ . Then it follows from property (a) of the measure  $\sigma_\alpha$  that  $\Delta$  contains infinitely many atoms of  $\sigma_\alpha$ . Since  $\sigma_\alpha$  is finite and  $\sigma_\alpha\{\xi\} = |\theta'(\xi)|^{-1}$  for every  $\xi \in a(\sigma_\alpha)$ , the function  $|\theta'|$  cannot be bounded on  $\Delta$ . This gives us condition (A1).

Condition (A2) is more delicate. To check it we need the following lemma.

**Lemma 2.2.** *Assume that the Clark measure  $\sigma_\alpha$  of an inner function  $\theta$  has properties (a) – (c). Then there exists a number  $\kappa > 0$  such that for every  $\xi \in a(\sigma_\alpha)$  the set  $D_\xi(\kappa) = \{z \in \mathbb{C} : |\xi - z| \leq \kappa \sigma_\alpha\{\xi\}\}$  is contained in  $\mathbb{D} \cup G_\theta$  and we have*

$$\frac{1}{2\sigma_\alpha\{\xi\}} \leq \left| \frac{\alpha - \theta(z)}{\xi - z} \right| \leq \frac{2}{\sigma_\alpha\{\xi\}} \quad (14)$$

for all  $z \in D_\xi(\kappa)$ .

**Proof.** Pick an atom  $\xi_0 \in a(\sigma_\alpha)$  and rewrite formula (12) in the following form:

$$\frac{1}{1 - \bar{\alpha}\theta(z)} = \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{1 - \xi z} + \frac{\sigma_\alpha\{\xi_0\}}{1 - \xi_0 z} + c_\alpha, \quad z \in \mathbb{D} \cup G_\theta. \quad (15)$$

We have

$$\left| \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{1 - \xi z} \right| \leq |(H_{\sigma_\alpha} 1)(\xi_0)| + \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{|\xi_0 - z| d\sigma_\alpha(\xi)}{|\xi - z| \cdot |\xi - \xi_0|}.$$

By property (c),  $|(H_{\sigma_\alpha} 1)(\xi_0)| \leq C_{\sigma_\alpha}$ . Put  $\kappa^* = (2B_{\sigma_\alpha})^{-1}$ . For  $\xi \in a(\sigma_\alpha) \setminus \{\xi_0\}$  and  $z \in D_{\xi_0}(\kappa^*)$  we have  $|\xi_0 - z| \leq \kappa^* \sigma_\alpha\{\xi_0\} \leq |\xi_0 - \xi|/2$  by property (b) of the measure  $\sigma_\alpha$ , which gives us the inequality  $|\xi - z| \geq |\xi - \xi_0| - |\xi_0 - z| \geq \frac{1}{2}|\xi - \xi_0|$ . It follows that for  $z \in D_{\xi_0}(\kappa^*)$  we have

$$\int_{\mathbb{T} \setminus \{\xi_0\}} \frac{|z - \xi_0| d\sigma_\alpha(\xi)}{|\xi - z| \cdot |\xi - \xi_0|} \leq 2\kappa^* \sigma_\alpha\{\xi_0\} \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{|\xi - \xi_0|^2}.$$

Denote by  $\Delta$  the closed arc of  $\mathbb{T}$  with endpoints  $\xi_{0\pm} \in a(\sigma_\alpha)$ . Using property (b), we obtain the estimate

$$\begin{aligned} \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{|\xi - \xi_0|^2} &\leq \int_{\mathbb{T} \setminus \Delta} \frac{d\sigma_\alpha(\xi)}{|\xi - \xi_0|^2} + \frac{2}{A_{\sigma_\alpha}^2 \sigma_\alpha\{\xi_0\}} \\ &\leq 2\pi B_{\sigma_\alpha} \int_{\mathbb{T} \setminus \Delta} \frac{dm(\xi)}{|\xi - \xi_0|^2} + \frac{2}{A_{\sigma_\alpha}^2 \sigma_\alpha\{\xi_0\}} \\ &\leq \frac{C_1}{\sigma_\alpha\{\xi_0\}}, \end{aligned} \quad (16)$$

where  $C_1$  is a constant depending only on the measure  $\sigma_\alpha$ . We now see from (15) that

$$\frac{1}{1 - \bar{\alpha}\theta(z)} = \frac{\sigma_\alpha\{\xi_0\}}{1 - \xi_0 z} + f_{\xi_0}(z), \quad z \in D_{\xi_0}(\kappa_1) \cap (\mathbb{D} \cup G_\theta), \quad (17)$$

where the function  $|f_{\xi_0}|$  is bounded by the constant  $C_2 = 2\kappa^* C_1 + C_{\sigma_\alpha} + |c_\alpha|$ . Take a number  $\kappa \leq \kappa^*$  such that  $C_2 \leq (2\kappa)^{-1}$ . We have  $D_{\xi_0}(\kappa) \subset D_{\xi_0}(\kappa^*)$  and

$$|f_{\xi_0}(z)| \leq \frac{1}{2} \left| \frac{\sigma_\alpha\{\xi_0\}}{1 - \xi_0 z} \right|$$

for all  $z \in D_{\xi_0}(\kappa) \cap (\mathbb{D} \cup G_\theta)$ . From here and (17) we get on  $D_{\xi_0}(\kappa) \cap (\mathbb{D} \cup G_\theta)$  the estimate

$$\frac{1}{2} \left| \frac{\sigma_\alpha\{\xi_0\}}{1 - \xi_0 z} \right| \leq \left| \frac{1}{1 - \bar{\alpha}\theta(z)} \right| \leq 2 \left| \frac{\sigma_\alpha\{\xi_0\}}{1 - \xi_0 z} \right|,$$

which shows that  $\theta$  admits the analytic continuation to a neighbourhood of  $D_{\xi_0}(\kappa)$  (that is,  $D_{\xi_0}(\kappa) \subset \mathbb{D} \cup G_\theta$ ) and proves formula (14) for points  $z \in D_{\xi_0}(\kappa)$ . Since

our choice of the number  $\kappa$  is uniform with respect to  $\xi_0 \in a(\sigma_\alpha)$ , the lemma is proved.  $\square$

*Notation.* In what follows we write  $E_1 \lesssim E_2$  (correspondingly,  $E_1 \gtrsim E_2$ ) for two expressions  $E_1, E_2$  to mean that there is a positive constant  $c_\theta$  depending only on the inner function  $\theta$  such that  $E_1 \leq c_\theta E_2$  (correspondingly,  $c_\theta E_1 \geq E_2$ ). We will write  $E_1 \asymp E_2$  if  $E_1 \lesssim E_2$  and  $E_1 \gtrsim E_2$ .

We are ready to complete the proof of Theorem 1. Differentiating (12) we get

$$\begin{aligned} \frac{\bar{\alpha}\theta'(z)}{(1-\bar{\alpha}\theta(z))^2} &= \int_{\mathbb{T}} \frac{\bar{\xi}d\sigma_\alpha(\xi)}{(1-\bar{\xi}z)^2}, \\ \frac{\bar{\alpha}\theta''(z)}{(1-\bar{\alpha}\theta(z))^2} + \frac{2\bar{\alpha}^2\theta'(z)^2}{(1-\bar{\alpha}\theta(z))^3} &= 2 \int_{\mathbb{T}} \frac{\bar{\xi}^2d\sigma_\alpha(\xi)}{(1-\bar{\xi}z)^3}. \end{aligned} \quad (18)$$

Pick a point  $\xi_0 \in a(\sigma_\alpha)$ . Let  $D_{\xi_0}(\kappa)$  be the set from Lemma 2.2. Denote by  $\partial D_{\xi_0}(\kappa)$  the boundary of  $D_{\xi_0}(\kappa)$ . By formula (14),  $|\alpha - \theta(z)| \geq \kappa/2$  on  $\partial D_{\xi_0}(\kappa)$ . Arguing as in the Lemma 2.2, from (18) we obtain the estimates

$$\begin{aligned} |\theta'(z)| &\lesssim \frac{\sigma_\alpha\{\xi_0\}}{|1-\bar{\xi}_0z|^2} + \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{|1-\bar{\xi}z|^2} \lesssim \frac{1}{\sigma_\alpha\{\xi_0\}}, \\ |\theta''(z)| &\lesssim |\theta'(z)|^2 + \frac{\sigma_\alpha\{\xi_0\}}{|1-\bar{\xi}_0z|^3} + \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{|1-\bar{\xi}z|^3} \lesssim \frac{1}{\sigma_\alpha\{\xi_0\}^2} \end{aligned} \quad (19)$$

for all  $z \in \partial D_{\xi_0}(\kappa)$ . By the maximum principle we have  $|\theta''(z)| \lesssim 1/\sigma_\alpha\{\xi_0\}^2$  for all points  $z \in D_{\xi_0}(\kappa)$ . On the unit circle  $\mathbb{T}$  we have

$$\frac{\bar{\xi}}{(1-\bar{\xi}z)^2} = \frac{-\bar{z}}{|1-\bar{\xi}z|^2}.$$

From here and formula (18) we get for  $z \in D_{\xi_0}(\kappa) \cap \mathbb{T}$  the estimate

$$|\theta'(z)| = \sigma_\alpha\{\xi_0\} \left| \frac{1-\bar{\alpha}\theta(z)}{1-\bar{\xi}_0z} \right|^2 + \int_{\mathbb{T} \setminus \{\xi_0\}} \left| \frac{1-\bar{\alpha}\theta(z)}{1-\bar{\xi}z} \right|^2 d\sigma_\alpha(\xi) \gtrsim \frac{1}{\sigma_\alpha\{\xi_0\}}. \quad (20)$$

Combining (19) and (20) we see that  $|\theta''/\theta'^2| \lesssim 1$  on  $D_{\xi_0}(\kappa) \cap \mathbb{T}$ . It remains to obtain the same estimate for points  $z \in \mathbb{T} \setminus \rho(\theta)$  that do not belong to the union of the sets  $D_\xi(\kappa)$ ,  $\xi \in a(\sigma_\alpha)$ , from Lemma 2.2. Take such a point  $z_0$ . We claim that  $|\theta(z_0) - \alpha| \geq \kappa/2$ . Indeed, assume the converse and find the connected component  $\Delta$  of the set  $\{\zeta \in \mathbb{T} \setminus \rho(\theta) : |\theta(\zeta) - \alpha| < \kappa/2\}$  containing the point  $z_0$ . Since the argument of the inner function  $\theta$  is monotonic on  $\Delta$ , there exists a point  $\xi \in \Delta$  such that  $\theta(\xi) = \alpha$ . By Lemma 2.1 we have  $\xi \in a(\sigma_\alpha)$ . Next, from (14) we see that  $|\alpha - \theta(z)| \geq \kappa/2$  for both points in  $\mathbb{T} \cap \partial D_\xi(\kappa)$ . Since  $\Delta$  is connected this yields the inclusion  $\Delta \subset D_\xi(\kappa)$  which gives us the contradiction with  $z_0 \notin D_\xi(\kappa)$ . Thus, we proved the inequality  $|\theta(z_0) - \alpha| \geq \kappa/2$ . Let  $\xi_{z_0}$  be the nearest point to  $z_0$  in  $a(\sigma_\alpha)$ . We have  $\kappa\sigma_\alpha\{\xi_{z_0}\} \leq |\xi_{z_0} - z| \leq B_{\sigma_\alpha}\sigma_\alpha\{\xi_{z_0}\}$ . These two estimates imply (19) and (20) for  $z = z_0$  and  $\xi_0 = \xi_{z_0}$ . It follows that  $|\theta''(z_0)/\theta'(z_0)^2| \lesssim 1$  and  $\theta$  satisfies condition (A2).  $\square$

**Remark.** Lemma 5.1 in [7] and formula (13) show that for every one-component inner function  $\theta$  there exist positive constants  $A_\theta, B_\theta, C_\theta$  such that  $A_\theta \leq A_{\sigma_\alpha}$ ,  $B_{\sigma_\alpha} \leq B_\theta$ ,  $C_{\sigma_\alpha} \leq C_\theta$  for all Clark measures  $\sigma_\alpha$  of  $\theta$ . Also, it follows from Lemma 5.1 in [7] that  $|\theta'(z)| \asymp 1/\sigma_\alpha\{\xi\}$  for all  $\xi \in a(\sigma_\alpha)$  and all  $z \in \mathbb{T}$  between the neighbours  $\xi_\pm$  of  $\xi$  in  $a(\sigma_\alpha)$ . In particular, we have  $\sigma_\beta(\Delta) \asymp \sigma_\alpha(\Delta) \asymp m(\Delta)$  for all  $\beta$



with  $|\beta| = 1$  and all arcs  $\Delta$  of the unit circle  $\mathbb{T}$  containing at least two atoms of the measure  $\sigma_\alpha$ .

### 3. PROOFS OF THEOREM 2 AND THEOREM 2'

We first prove Theorem 2'. The following result is classical, for the proof see [11] or Chapter 9 in [10].

**Theorem.** (*D. N. Clark*) *Let  $\theta$  be an inner function and let  $\sigma_\alpha$  be its Clark measure. The natural embedding  $V_\alpha : K_\theta^2 \hookrightarrow L^2(\sigma_\alpha)$  defined on the reproducing kernels of the space  $K_\theta^2$  by  $V_\alpha \left( \frac{1-\overline{\theta(\lambda)}\theta}{1-\lambda z} \right) = \frac{1-\overline{\theta(\lambda)}\alpha}{1-\lambda z}$  can be extended to the whole space  $K_\theta^2$  as the unitary operator from  $K_\theta^2$  to  $L^2(\sigma_\alpha)$ . For every  $f \in L^2(\sigma_\alpha)$  the function*

$$F(z) = \int_{\mathbb{T}} f(\xi) \frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} d\sigma_\alpha(\xi) \quad (21)$$

*in the unit disk  $\mathbb{D}$  belongs to  $K_\theta^2$  and  $V_\alpha F = f$  as elements in  $L^2(\sigma_\alpha)$ .*

It worth be mentioned that A. G. Poltoratski [22] established the existence of angular boundary values  $\sigma_\alpha$ -almost everywhere on  $\mathbb{T}$  for all functions in the space  $K_\theta^2$ , thus proving that the unitary operator  $V_\alpha$  in Clark theorem acts as the natural embedding on the whole space  $K_\theta^2$ . In our situation this follows from a very simple argument, see Lemma 3.3 in Section 3.2.

The embedding  $V_\alpha : K_\theta^p \hookrightarrow L^p(\sigma_\alpha)$  defined on the linear span of the reproducing kernels of  $K_\theta^p$  might be unbounded for  $1 \leq p < 2$  and might have the unbounded inverse  $V_\alpha^{-1} : L^p(\sigma_\alpha) \hookrightarrow K_\theta^p$  for  $2 < p \leq \infty$ , see Section 3 in [2]. However, the situation is ideal for the one-component inner functions  $\theta$ , as following results show:

- $V_\alpha K_\theta^p \subset L^p(\sigma_\alpha)$  for  $1 < p < \infty$  – A. L. Volberg, S. R. Treil [26];
- $V_\alpha K_\theta^p = L^p(\sigma_\alpha)$  for  $1 < p < \infty$  – A. B. Aleksandrov [2];
- $V_\alpha K_\theta^p \subset L^p(\sigma_\alpha)$  for  $0 < p \leq 1$  – A. B. Aleksandrov [3].

Theorem 2' says that  $V_\alpha K_\theta^1 = H_{at}^1(\sigma_\alpha)$  for every one-component inner function  $\theta$ . We are ready to prove its easy part – the inclusion  $V_\alpha K_\theta^1 \supset H_{at}^1(\sigma_\alpha)$ .

**3.1. Proof of the part “ $\Rightarrow$ ” in Theorem 2'.** Let  $\mu$  be a measure on the unit circle  $\mathbb{T}$  with properties (a) – (c). Take a complex number  $\alpha$  of unit modulus and construct the one-component inner function  $\theta$  with the Clark measure  $\sigma_\alpha = \mu$ . We want to show that every function  $f \in H_{at}^1(\sigma_\alpha)$  admits the analytic continuation to the open unit disk  $\mathbb{D}$  as a function  $F \in K_\theta^1 \cap zH^1$  with  $\|F\|_{L^1(\mathbb{T})} \lesssim \|f\|_{H_{at}^1(\sigma_\alpha)}$ . At first, assume that  $f$  is a  $\sigma_\alpha$ -atom supported on an arc  $\Delta \subset \mathbb{T}$  with center  $\xi_c$ . Then  $f \in L^2(\sigma_\alpha)$  and the function  $F$  in formula (21) lies in the space  $K_\theta^2 \subset K_\theta^1$  by Clark theorem. Since  $\int_{\mathbb{T}} f d\sigma_\alpha = 0$ , we have  $F(0) = 0$ . Moreover, we see from Lemma 2.1 that  $F(\xi) = f(\xi)$  for all  $\xi \in a(\sigma_\alpha)$ . Let us check that the norm of  $F$  in  $L^1(\mathbb{T})$  is bounded by a constant depending only on the inner function  $\theta$ . By Aleksandrov disintegration theorem (see [1] or Section 9.4 in [10]), we have

$$\int_{\mathbb{T}} |F| dm = \int_{\mathbb{T}} \int_{\mathbb{T}} |V_\beta F(\xi)| d\sigma_\beta(\xi) dm(\beta). \quad (22)$$

Fix a complex number  $\beta \neq \alpha$  of unit modulus. We claim that  $\|V_\beta F\|_{L^1(\sigma_\beta)} \lesssim 1$ . Denote by  $2\Delta$  the arc of  $\mathbb{T}$  with center  $\xi_c$  such that  $m(2\Delta) = 2m(\Delta)$  (in the case

where  $m(\Delta) \geq 1/2$  put  $2\Delta = \mathbb{T}$ ). Break the integral  $\int_{\mathbb{T}} |V_\beta F| d\sigma_\beta$  into two parts,

$$\int_{\mathbb{T}} |V_\beta F(\xi)| d\sigma_\beta(\xi) = \int_{2\Delta} |V_\beta F(\xi)| d\sigma_\beta(\xi) + \int_{\mathbb{T} \setminus 2\Delta} |V_\beta F(\xi)| d\sigma_\beta(\xi). \quad (23)$$

By Clark theorem we have  $\|V_\beta F\|_{L^2(\sigma_\beta)} = \|F\|_{L^2(\mathbb{T})} = \|V_\alpha F\|_{L^2(\sigma_\alpha)}$ . Moreover, we have  $\|V_\alpha F\|_{L^2(\sigma_\alpha)} \leq 1/\sqrt{\sigma_\alpha(\Delta)}$  because the function  $V_\alpha F = f$  is a  $\sigma_\alpha$ -atom supported on the arc  $\Delta$ . This yields the inequality

$$\int_{2\Delta} |V_\beta F(\xi)| d\sigma_\beta(\xi) \leq \sqrt{\sigma_\beta(2\Delta)} \cdot \|V_\beta F\|_{L^2(\sigma_\beta)} \leq \sqrt{\sigma_\beta(2\Delta)/\sigma_\alpha(\Delta)}. \quad (24)$$

Note that the arc  $\Delta$  contains at least two points in  $a(\sigma_\alpha)$  because  $f$  has zero  $\sigma_\alpha$ -mean on  $\Delta$ . Hence  $\sigma_\alpha(\Delta) \asymp m(\Delta)$  and  $\sigma_\beta(2\Delta) \asymp m(2\Delta)$ , see remark after the proof of Theorem 1. This shows that  $\int_{2\Delta} |V_\beta F(\xi)| d\sigma_\beta(\xi) \lesssim 1$ . Let us now estimate the second term in (23). Take a point  $z \in a(\sigma_\beta) \setminus 2\Delta$ . Using the fact that  $f$  is a  $\sigma_\alpha$ -atom we obtain the estimate

$$\begin{aligned} |V_\beta F(z)| &= \left| \int_{\Delta} f(\xi) \frac{1 - \bar{\alpha}\beta}{1 - \bar{\xi}z} d\sigma_\alpha(\xi) \right| \\ &= \left| \int_{\Delta} f(\xi) \left( \frac{1 - \bar{\alpha}\beta}{1 - \bar{\xi}z} - \frac{1 - \bar{\alpha}\beta}{1 - \bar{\xi}_c z} \right) d\sigma_\alpha(\xi) \right| \\ &\leq 2 \int_{\Delta} |f(\xi)| \left| \frac{\xi - \xi_c}{(1 - \bar{\xi}z)(1 - \bar{\xi}_c z)} \right| d\sigma_\alpha(\xi) \\ &\leq \frac{2\pi m(\Delta)}{|z - \xi_c|^2} \cdot \sup_{\xi \in \Delta} \left| \frac{z - \xi_c}{z - \xi} \right| \cdot \int_{\Delta} |f(\xi)| d\sigma_\alpha(\xi) \\ &\leq \frac{4\pi m(\Delta)}{|z - \xi_c|^2}. \end{aligned} \quad (25)$$

From here we get

$$\int_{\mathbb{T} \setminus 2\Delta} |V_\beta F(z)| \lesssim m(2\Delta) \cdot \int_{\mathbb{T} \setminus 2\Delta} \frac{d\sigma_\beta(z)}{|z - \xi_c|^2} \lesssim 1. \quad (26)$$

Hence the norm of  $F$  in  $L^1(\mathbb{T})$  is bounded by a constant depending only on  $\theta$ . Now take an arbitrary function  $f \in H_{at}^1(\sigma_\alpha)$  and consider its representation  $f = \sum \lambda_k f_k$ , where  $f_k$  are  $\sigma_\alpha$ -atoms and  $\sum_k |\lambda_k| \leq 2\|f\|_{H_{at}^1(\sigma_\alpha)}$ . Let  $F_k$  be the functions in  $K_\theta^2$  such that  $V_\alpha F_k = f_k$ . Then the sum  $\sum \lambda_k F_k$  converges absolutely in  $L^1(\mathbb{T})$  to a function  $F \in K_\theta^1$  and we have  $\|F\|_{L^1(\mathbb{T})} \lesssim \|f\|_{H_{at}^1(\sigma_\alpha)}$ . From formula (21) we get

$$F(z) = \sum_k \lambda_k \int_{\mathbb{T}} f_k(\xi) \frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} d\sigma_\alpha(\xi) = \int_{\mathbb{T}} f(\xi) \frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.$$

Since  $f \in L^1(\mathbb{T})$ , this formula determines the analytic continuation of  $F$  to the domain  $\mathbb{D} \cup G_\theta$ . From Lemma 2.1 we see that  $F(\xi) = f(\xi)$  for all  $\xi \in a(\sigma_\alpha)$ .  $\square$

**3.2. Preliminaries for the proof of the part “ $\Leftarrow$ ” in Theorem 2’.** Let  $\theta$  be a one-component inner function with the Clark measure  $\sigma_\alpha$ . Introduce positive constants  $A_{\sigma_\alpha}, B_{\sigma_\alpha}$  such that

$$\tilde{A}_{\sigma_\alpha} m[\xi, \xi_\pm] \leq \sigma_\alpha\{\xi\} \leq \tilde{B}_{\sigma_\alpha} m[\xi, \xi_\pm], \quad \xi \in a(\sigma_\alpha).$$

Here  $[\xi, \xi_-], [\xi, \xi_+]$  are the closed arcs of  $\mathbb{T}$  with endpoints  $\xi, \xi_\pm \in a(\sigma_\alpha)$  such that the corresponding open arcs  $(\xi, \xi_\pm)$  do not intersect  $\text{supp } \sigma_\alpha$ . Take a positive

number  $\kappa \leq (2\tilde{B}_{\sigma_\alpha})^{-1}$  for which estimate (14) holds true. Denote by  $D_{\sigma_\alpha}(\kappa)$  the union of the sets  $D_\xi(\kappa)$ ,  $\xi \in a(\sigma_\alpha)$  from Lemma 2.2.

**Lemma 3.1.** *For every arc  $\Delta$  of  $\mathbb{T}$  containing at least one atom of the measure  $\sigma_\alpha$  we have  $m(\Delta) \leq (2/\tilde{A}_{\sigma_\alpha})\sigma_\alpha(\Delta)$ . If  $\Delta$  contains two or more atoms of  $\sigma_\alpha$ , we have  $\sigma_\alpha(\Delta) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(\kappa))$ . In particular, the sets  $D_\xi(\kappa)$ ,  $\xi \in a(\sigma_\alpha)$  are disjoint.*

**Proof.** It is sufficient to prove the statement in the case where  $\Delta$  contains only finite number of atoms of  $\sigma_\alpha$ . Enumerate the atoms clockwise:  $\xi_1, \dots, \xi_n$ . Find the neighbours of  $\xi_1, \xi_n$  in  $a(\sigma_\alpha) \setminus \Delta$  and denote them by  $\xi_0$  and  $\xi_{n+1}$ , correspondingly. We have

$$m(\Delta) \leq \sum_{k=0}^n m[\xi_k, \xi_{k+1}] \leq \frac{2}{\tilde{A}_{\sigma_\alpha}} \sum_{k=1}^n \sigma_\alpha\{\xi_k\} = \frac{2}{\tilde{A}_{\sigma_\alpha}} \sigma_\alpha(\Delta).$$

In the case where  $n \geq 2$  we have

$$\sigma_\alpha(\Delta) = \sum_{k=1}^n \sigma_\alpha\{\xi_k\} \leq 2\tilde{B}_{\sigma_\alpha}m(\Delta) \leq 2\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(\kappa)) + \tilde{B}_{\sigma_\alpha}\kappa\sigma_\alpha(\Delta).$$

Now use the assumption  $\kappa \leq (2\tilde{B}_{\sigma_\alpha})^{-1}$  and get  $\sigma_\alpha(\Delta) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(\kappa))$ .  $\square$

**Lemma 3.2.** *There exists  $\varepsilon > 0$  such that  $|\alpha - \theta(z)| \geq \varepsilon$  for all  $z \in \mathbb{D} \setminus D_{\sigma_\alpha}(\kappa)$ .*

**Proof.** Let  $\delta \in (0, 1)$  be a number such that the set  $\Omega_\delta = \{z \in \mathbb{D} : |\theta(z)| < 1\}$  is connected. The set

$$\Omega_{\delta, \frac{1}{\delta}} = \{z \in \mathbb{D} \cup G_\theta : \delta < |\theta(z)| < 1/\delta\}$$

is at most countable union of the open connected components,  $\mathcal{O}_k$ . It was proved by B. Cohn [12] that the restriction of the inner function  $\theta$  to each of the sets  $\mathcal{O}_k$  is a covering map from  $\mathcal{O}_k$  to the ring  $R_\delta = \{z \in \mathbb{C} : \delta < |z| < 1/\delta\}$ . Take a positive number  $\varepsilon < \min(\kappa/2, 1 - \delta)$ . We claim that every connected component  $E$  of the set  $L_\varepsilon = \{z \in \mathbb{D} \cup G_\theta : |\alpha - \theta(z)| < \varepsilon\}$  contains an atom of  $\sigma_\alpha$ . Indeed, we have  $E \subset \mathcal{O}_k$  for some index  $k$  because  $L_\varepsilon \subset \Omega_{\delta, \frac{1}{\delta}}$ . Since  $\theta$  is a covering map from  $\mathcal{O}_k$  to  $R_\delta$ , there exists a number  $\varepsilon_1$  (which can be taken to be less than  $\varepsilon$ ) such that the preimage of  $\{\zeta : |\alpha - \zeta| < \varepsilon_1\}$  under  $\theta$  on  $\mathcal{O}_k$  is at most countable union of the open disjoint sets  $\mathcal{O}_{km} \subset \mathcal{O}_k$  and  $\theta$  is a homeomorphism from  $\mathcal{O}_{km}$  to  $\{\zeta : |\zeta - \alpha| < \varepsilon_1\}$  for every  $m$ . By the minimum principle,  $\inf_{z \in E} |\theta(z) - \alpha| = 0$ . It follows that  $E \cap \mathcal{O}_{km} \neq \emptyset$  for some index  $m$ . Since  $E$  is connected and  $|\theta - \alpha| < \varepsilon_1 < \varepsilon$  on  $\mathcal{O}_{km}$ , we have  $\mathcal{O}_{km} \subset E$ . But every set  $\mathcal{O}_{km}$  contains the unique point  $\xi$  with  $\theta(\xi) = \alpha$ . By Lemma 2.1,  $\xi \in a(\sigma_\alpha)$  and thus  $E \cap a(\sigma_\alpha) \neq \emptyset$ . To prove the lemma it is sufficient to show that  $E \subset D_\xi(\kappa)$ . For every  $z \in E \cap D_\xi(\kappa)$  we get from (14) the estimate

$$|\xi - z| \leq 2|\alpha - \theta(z)|\sigma_\alpha\{\xi\} \leq 2\varepsilon\sigma_\alpha\{\xi\}.$$

Hence  $E$  does not intersect the circle  $\{z \in \mathbb{C} : |z - \xi| = r\sigma_\alpha\{\xi\}\}$  for every  $r \in (2\varepsilon, \kappa)$ . Since the set  $E$  is connected this yields the desired inclusion  $E \subset D_\xi(\kappa)$ .  $\square$

**Lemma 3.3.** *Let  $\theta$  be an inner function with  $\rho(\theta) \neq \mathbb{T}$ . Then every function in  $K_\theta^1$  admits the analytic continuation from the unit disk  $\mathbb{D}$  to the domain  $\mathbb{D} \cup G_\theta$ . Consequently, if  $\sigma_\alpha(\rho(\theta)) = 0$  for a Clark measure  $\sigma_\alpha$  of  $\theta$ , then every function in  $K_\theta^1$  has a trace on the set  $a(\sigma_\alpha) \subset \mathbb{D} \cup G_\theta$  of full measure  $\sigma_\alpha$ .*

**Proof.** For every function  $F \in K_\theta^1$  we have  $\bar{\theta}F \in \overline{zH^1}$  on  $\mathbb{T}$ . Hence,

$$F(z) = \int_{\mathbb{T}} F(\xi) \frac{1 - \theta(z)\overline{\theta(\xi)}}{1 - z\bar{\xi}} dm(\xi), \quad z \in \mathbb{D}. \quad (27)$$

Extend the inner function  $\theta$  to the domain  $\mathbb{D} \cup G_\theta$  by formula (10). The right hand side of (27) then determines the analytic continuation of the function  $F$  to  $\mathbb{D} \cup G_\theta$ . By Lemma 2.1 we have  $\rho(\sigma_\alpha) = \rho(\theta)$  which completes the proof.  $\square$

**Lemma 3.4.** *Let  $\theta$  be an inner function and let  $G \in K_\theta^1 \cap zH^1$ . Then there exist functions  $G_1, G_2 \in K_\theta^1 \cap zH^1$  such that  $G = G_1 + iG_2$  and  $G_{1,2} = \theta\bar{G}_{1,2}$  on  $\mathbb{T} \setminus \rho(\theta)$ . Moreover, we have  $\|G_{1,2}\|_{L^1(\mathbb{T})} \leq \|G\|_{L^1(\mathbb{T})}$ .*

**Proof.** Consider the function  $\tilde{G} = \theta\bar{G}$  on the unit circle  $\mathbb{T}$ . We have

$$\tilde{G} \in \theta(\overline{zH^1} \cap z\bar{\theta}H^1) = \bar{z}\theta\bar{H^1} \cap zH^1 = K_\theta^1 \cap zH^1.$$

This shows that  $G$  can be continued to the open unit disk  $\mathbb{D}$  as a function from the space  $K_\theta^1 \cap zH^1$ . Now put  $G_1 = (G + \tilde{G})/2$ ,  $G_2 = (G - \tilde{G})/2i$  and obtain the desired representation.  $\square$

**3.3. Proof of the part “ $\Leftarrow$ ” in Theorem 2’.** Let  $\mu$  be a measure on the unit circle  $\mathbb{T}$  with properties (a) – (c) and let  $|\alpha| = 1$ . Consider the one-component inner function  $\theta$  with the Clark measure  $\sigma_\alpha = \mu$ . Take a function  $F \in K_\theta^1 \cap zH^1$ . By Lemma 3.3,  $F$  is analytic on the domain  $\mathbb{D} \cup G_\theta$ . Denote by  $f$  its trace on the set  $a(\sigma_\alpha) \subset \mathbb{D} \cup G_\theta$  of full measure  $\sigma_\alpha$ . Our aim is to prove that  $f \in H_{at}^1(\sigma_\alpha)$  and  $\|f\|_{H_{at}^1(\sigma_\alpha)} \lesssim \|F\|_{L^1(\mathbb{T})}$ . At first, assume that  $F \in K_\theta^2 \cap zH^2$  and  $F = \theta\bar{F}$  on  $\mathbb{T} \setminus \rho(\theta)$ . We will need the following modification of the Lusin-Privalov construction (see Section III.D in [16] for the standard one). Consider the non-tangential maximal function of  $F$ ,

$$F^*(\xi) = \sup_{z \in \Lambda_\xi} |F(z)|, \quad \xi \in \mathbb{T},$$

where  $\Lambda_\xi$  denotes the convex hull of the set  $\{\xi\} \cup \{z \in \mathbb{D} : |z| \leq 1/\sqrt{2}\}$ . Put

$$S_F(\lambda) = \overline{\mathbb{D}} \setminus \{z \in \overline{\mathbb{D}} : z \in \Lambda_\xi \text{ for some } \xi \in \mathbb{T} \text{ with } F^*(\xi) < \lambda\}.$$

Let  $D_{\sigma_\alpha}(\kappa)$  be the set defined at the beginning of Section 3.2. By Lemma 3.2, we have  $|\alpha - \theta| \geq \varepsilon$  on  $\mathbb{D} \setminus D_{\sigma_\alpha}(\kappa)$ . Denote by  $R_F(\lambda)$  the union of those connected components of the set  $S_F(\lambda) \cup D_{\sigma_\alpha}(\kappa)$  for which we have  $E \cap S_F(\lambda) \neq \emptyset$  and  $E \cap D_{\sigma_\alpha}(\kappa) \neq \emptyset$ . The sets  $R_F(\lambda)$  are closed and have the following properties:

- (1) If  $\lambda_1 < \lambda_2$ , then  $R_F(\lambda_2) \subset R_F(\lambda_1)$ ;
- (2)  $|F(z)| \leq \lambda$  for  $\sigma_\alpha$ -almost all points  $z \in \mathbb{T} \setminus R_F(\lambda)$ ;
- (3)  $|F(z)| \leq \lambda$  and  $|\alpha - \theta(z)| \geq \varepsilon$  for  $z \in \partial R_F(\lambda) \cap \mathbb{D}$ .

More special properties of the sets  $R_F(\lambda)$  are collected in the following lemma.

**Lemma 3.5.** *Let  $E$  be a connected component of the set  $R_F(\lambda)$ . Put  $\gamma = \partial E \cap \mathbb{D}$  and  $\Delta = \partial E \cap \mathbb{T}$ . There exist constants  $c_4, c_5, c_6$  depending only on  $\theta$  such that*

- (4)  $\gamma$  is a rectifiable curve with length  $|\gamma| \leq c_4\sigma_\alpha(\Delta)$ ;
- (5)  $\sigma_\alpha(\Delta) \leq c_5m(\Delta \cap S_F(\lambda))$  if  $E$  contains at least two atoms of  $\sigma_\alpha$ ;
- (6)  $\frac{1}{\sigma_\alpha(\Delta)} \left| \int_\Delta f d\sigma_\alpha \right| \leq c_6\lambda$ .

One can take  $c_4 = 40/\tilde{A}_{\sigma_\alpha}$ ,  $c_5 = 4\tilde{B}_{\sigma_\alpha}$ ,  $c_6 = 60/(\varepsilon\tilde{A}_{\sigma_\alpha})$ .

**Proof.** By the construction and Lemma 3.1 we have

$$|\gamma| \leq (\sqrt{2} + \pi/2)|\Delta| \leq 20m(\Delta) \leq 40\sigma_\alpha(\Delta)/\tilde{A}_{\sigma_\alpha}.$$

In the case where the arc  $\Delta$  contains at least two atoms of the measure  $\sigma_\alpha$  Lemma 3.1 gives us the estimate

$$\sigma_\alpha(\Delta) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(\kappa)) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \cap S_F(\lambda)).$$

Let us check property (6). At first, assume that  $\gamma \cap \rho(\theta) = \emptyset$ . Then we have  $\gamma \cap \text{supp } \sigma_\alpha = \emptyset$  by the construction. For  $z \in \mathbb{C}$  with  $|z| \geq 1$  denote  $z^* = 1/\bar{z}$  and put  $\gamma^* = \{z \in \mathbb{C} : z^* \in \gamma\}$ . The set  $\Gamma = \gamma \cup \gamma^*$  is a rectifiable curve in  $\mathbb{C}$  with length  $|\Gamma| \leq 3|\gamma|$ . Let us check that

$$\left| \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} \right| \leq 2\varepsilon^{-1}\lambda, \quad z \in \Gamma \cap (\mathbb{D} \cup G_\theta). \quad (28)$$

For  $z \in \gamma$  we have  $|z| \geq 1/\sqrt{2}$ ,  $|F| \leq \lambda$ ,  $|\alpha - \theta| \geq \varepsilon$  and therefore (28) holds. The function  $z \mapsto \overline{F(z^*)/\theta(z^*)}$  is analytic on the interior of  $G_\theta$  and coincides with the function  $F$  on  $G_\theta \cap \mathbb{T} = \mathbb{T} \setminus \rho(\theta)$  (recall that  $F$  admits the analytic continuation to the domain  $\mathbb{D} \cup G_\theta$  by Lemma 3.3 and  $F = \theta\bar{F}$  on  $\mathbb{T} \setminus \rho(\theta)$  by the assumption). By the uniqueness of the analytic continuation we have  $F(z) = \overline{F(z^*)/\theta(z^*)}$  for all  $z \in G_\theta$ . Now take a point  $z \in G_\theta$  and compute

$$\frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} = \frac{\overline{z^*F(z^*)/\theta(z^*)}}{1 - \bar{\alpha}/\theta(z^*)} = \frac{\overline{z^*F(z^*)}}{\theta(z^*) - \alpha}.$$

This yields estimate (28) for  $z \in \gamma^* \cap G_\theta$ . Next, we claim that

$$\int_\Delta f(\xi) d\sigma_\alpha(\xi) = -\frac{1}{2\pi i} \oint_\Gamma \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} dz. \quad (29)$$

Indeed, using formula (21) for the function  $F/z \in K_\theta^2$  we obtain

$$\begin{aligned} \oint_\Gamma \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} dz &= \oint_\Gamma \frac{1}{1 - \bar{\alpha}\theta(z)} \int_{\mathbb{T}} \bar{\xi} f(\xi) \frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} dz d\sigma_\alpha(\xi) \\ &= \int_{\mathbb{T}} f(\xi) \oint_\Gamma \frac{1}{\xi - z} dz d\sigma_\alpha(\xi) = -2\pi i \int_{\mathbb{T}} f(\xi) \chi_\Delta(\xi) d\sigma_\alpha(\xi), \end{aligned} \quad (30)$$

where  $\chi_\Delta$  denotes the indicator of the set  $\Delta$ . Note that change of the order of integration is possible because  $\Gamma \cap \text{supp } \sigma_\alpha = \emptyset$  and therefore all integrals in (30) are absolutely convergent. We now see from (28) and (29) that

$$\left| \int_\Delta f(\xi) d\sigma_\alpha(\xi) \right| \leq (\pi\varepsilon)^{-1}\lambda|\Gamma| \leq 3(\pi\varepsilon)^{-1}\lambda|\gamma| \leq 3(\pi\varepsilon)^{-1}c_3 \cdot \lambda \cdot \sigma_\alpha(\Delta).$$

This gives us property (5) in the case where  $\gamma \cap \rho(\theta) = \emptyset$ . The general case can be reduced to just considered one by a small perturbation of the contour  $\gamma$ ; use the fact that  $f \in L^2(\sigma_\alpha)$  by Clark theorem and property (a) of the measure  $\sigma_\alpha$  from Theorem 1.  $\square$

Lemma 3.5 is the key argument in the proof of Theorem 2'. The rest of the proof is a standard Calderón-Zigmund decomposition. We will follow the exposition in

Section VII.E of [16]. For each  $\lambda > 0$  the set  $\Delta_F(\lambda) = R_F(\lambda) \cap \mathbb{T}$  is a union of closed disjoint arcs  $\Delta_F^k(\lambda)$ ,  $\Delta_F(\lambda) = \cup_{k \in I_\lambda} \Delta_F^k(\lambda)$ . Consider the functions

$$G_\lambda = \begin{cases} f, & \xi \in \mathbb{T} \setminus \Delta_F(\lambda), \\ \langle f \rangle_{\Delta_F^k(\lambda), \sigma_\alpha}, & \xi \in \Delta_F^k(\lambda), \end{cases} \quad B_\lambda = \begin{cases} 0, & \xi \in \mathbb{T} \setminus \Delta_F(\lambda), \\ f - \langle f \rangle_{\Delta_F^k(\lambda), \sigma_\alpha}, & \xi \in \Delta_F^k(\lambda). \end{cases}$$

By Lemma 3.5 we have  $|G_\lambda| \leq c_6 \lambda$   $\sigma_\alpha$ -almost everywhere on  $\mathbb{T}$ . The function  $B_\lambda$  has zero  $\sigma_\alpha$ -mean on each arc  $\Delta_F^k(\lambda)$ ,  $k \in I_\lambda$ . For every integer  $n \in \mathbb{Z}$  set  $g_n = G_{2^n}$  and  $b_n = B_{2^n}$ . Fix a number  $N_0 \in \mathbb{Z}$  such that

$$2^{N_0} < \inf_{|z| \leq 1/\sqrt{2}} |F(z)| \leq 2^{N_0+1}.$$

Note that  $\Delta_F(2^{N_0}) = \mathbb{T}$ . By formula (21),

$$g_{N_0} = \frac{1}{\sigma_\alpha(\mathbb{T})} \int_{\mathbb{T}} f d\sigma_\alpha = \frac{F(0)}{\sigma_\alpha(\mathbb{T})(1 - \bar{\alpha}\theta(0))} = 0.$$

Since  $f$  is finite at each point  $\xi \in a(\sigma_\alpha)$  we have  $f(\xi) = g_N(\xi)$  for every sufficiently big number  $N$ . Hence

$$f(\xi) = \sum_{n=N_0}^{\infty} (g_{n+1}(\xi) - g_n(\xi)), \quad \xi \in a(\sigma_\alpha), \quad (31)$$

where the sum converges pointwise (in fact, only finite number of summands in (31) are non-zero for every  $\xi \in a(\sigma_\alpha)$ ). Note that  $f = b_n + g_n$  and  $g_{n+1} - g_n = b_n - b_{n+1}$  for all  $n \geq N_0$ . Let  $I'_{2^n}$  be the set of indexes  $k \in I_{2^n}$  such that the set  $\Delta_F^k(2^n)$  contains at least two atoms of the measure  $\sigma_\alpha$ . The function  $g_{n+1} - g_n$  vanishes  $\sigma_\alpha$ -almost everywhere on each of the sets  $\Delta_F^k(2^n)$ ,  $k \in I_{2^n} \setminus I'_{2^n}$ . Indeed, for such index  $k$  we have by the construction. Hence  $g_n(\xi) = g_{n+1}(\xi) = f(\xi)$  because the  $\sigma_\alpha$ -mean of  $f$  on any arc containing the only point  $\xi \in a(\sigma_\alpha)$  equals  $f(\xi)$ . Define

$$\tilde{a}_{n,k} = \chi_{\Delta_F^k(2^n)}(b_n - b_{n+1}), \quad n \geq N_0, \quad k \in I'_{2^n},$$

where  $\chi_{\Delta_F^k(2^n)}$  is the indicator of the set  $\Delta_F^k(2^n)$ . The functions  $\tilde{a}_{n,k}$  have zero  $\sigma_\alpha$ -mean on  $\mathbb{T}$ . Indeed, let  $I$  denote the set of indexes  $m$  such that  $\Delta_F^m(2^{n+1}) \subset \Delta_F^k(2^n)$  (note that  $\Delta_F(2^{n+1}) \subset \Delta_F(2^n)$  by property (1) of the sets  $R_F(\lambda)$ ). Then

$$\int_{\mathbb{T}} \tilde{a}_{n,k} d\sigma_\alpha = \int_{\Delta_F^k(2^n)} (b_n - b_{n+1}) d\sigma_\alpha = - \sum_{m \in I} \int_{\Delta_F^m(2^{n+1})} b_{n+1} d\sigma_\alpha = 0.$$

Also, we have  $|\tilde{a}_{n,k}| \leq |g_n| + |g_{n+1}| \leq 3c_6 \cdot 2^n$  on  $\mathbb{T}$  for every  $n \geq N_0$  and  $k \in I'_{2^n}$ . Now put

$$a_{n,k} = \frac{\tilde{a}_{n,k}}{3c_6 \cdot 2^n \cdot \sigma_\alpha(\Delta_F^k(2^n))}, \quad n \geq N_0, \quad k \in I'_{2^n},$$

and observe that  $a_{n,k}$  are atoms with respect to the measure  $\sigma_\alpha$ . It follows from formula (31) that

$$f(\xi) = \sum_{n \geq N_0} \sum_{k \in I'_{2^n}} \lambda_{n,k} a_{n,k}(\xi), \quad \xi \in a(\sigma_\alpha), \quad (32)$$

where  $\lambda_{n,k} = 3c_6 \cdot 2^n \cdot \sigma_\alpha(\Delta_F^k(2^n))$  and the sum is convergent pointwise. Since the set  $a(\sigma_\alpha)$  has full measure  $\sigma_\alpha$  it remains to check that

$$\sum_{n \geq N_0} \sum_{k \in I'_{2^n}} \lambda_{n,k} \lesssim \|F\|_{L^1(\mathbb{T})}. \quad (33)$$

By Lemma 3.5 we have

$$\sigma_\alpha(\Delta_F^k(2^n)) \leq c_5 m(\Delta_F^k(2^n) \cap S_F(2^n))$$

for every  $n \geq N_0$  and  $k \in I'_{2n}$ . Hence,

$$\begin{aligned} \sum_{n \geq N_0} \sum_{k \in I'_{2n}} 2^n \sigma_\alpha(\Delta_F^k(2^n)) &\leq c_5 \sum_{n \geq N_0} \sum_{k \in I'_{2n}} 2^n m(\Delta_F^k(2^n) \cap S_F(2^n)) \\ &\leq c_5 \sum_{n \geq N_0} 2^n m(S_F(2^n) \cap \mathbb{T}) = c_5 \sum_{n \geq N_0} 2^n m(\{\xi \in \mathbb{T} : F^*(\xi) \geq 2^n\}). \end{aligned}$$

The last sum does not exceed

$$\begin{aligned} \sum_{n \geq N_0} \sum_{l \geq 0} 2^n m(\{\xi \in \mathbb{T} : 2^{n+l} \leq F^*(\xi) < 2^{n+l+1}\}) &\leq \\ &\leq \sum_{l \geq 0} m(\{\xi \in \mathbb{T} : 2^{N_0+l} \leq F^*(\xi) < 2^{N_0+l+1}\}) \sum_{k=N_0}^l 2^{N_0+k} \\ &\leq \sum_{l \geq 0} 2^{N_0+l+1} \cdot m(\{\xi \in \mathbb{T} : 2^{N_0+l} \leq F^*(\xi) < 2^{N_0+l+1}\}) \\ &\leq 2 \|F^*\|_{L^1(\mathbb{T})} \leq 2M \|F\|_{L^1(\mathbb{T})}, \end{aligned}$$

where  $M$  denotes the norm of the maximal operator  $F \mapsto F^*$  on  $H^1$ . Thus, inequality (33) holds with the constant  $6c_5c_6M$  and formula (32) gives us the atomic decomposition of the trace  $f$  provided  $F \in K_\theta^2 \cap zH^2$  and  $F = \theta \bar{F}$ . Now consider arbitrary function  $F \in K_\theta^1 \cap zH^1$  with the trace  $f$  on the set  $a(\sigma_\alpha)$ . Since  $K_\theta^2 \cap zH^2$  is the dense subset of  $K_\theta^1 \cap zH^1$  in norm of  $L^1(\mathbb{T})$  one can find functions  $F_k \in K_\theta^2 \cap zH^2$  such that  $F = \sum_k F_k$  and  $\|F\|_{L^1(\mathbb{T})} \geq \frac{1}{2} \sum_k \|F_k\|_{L^1(\mathbb{T})}$ . Let  $G_{1,k}, G_{2,k}$  be the functions from Lemma 3.4 for  $G = F_k$  and let  $g_{1,k}, g_{2,k}$  be their traces on  $a(\sigma_\alpha)$ . We have  $f(\xi) = \sum g_{1,k}(\xi) + i \sum g_{2,k}(\xi)$  for every  $\xi \in a(\sigma_\alpha)$ , see formula (27). It follows from the first part of the proof that  $f$  admits the atomic decomposition with respect to the measure  $\sigma_\alpha$  and we have  $\|f\|_{H_{at}^1(\sigma_\alpha)} \leq 24c_5c_6M \|F\|_{L^1(\mathbb{T})}$ .  $\square$

**3.4. Proof of Theorem 2.** Since  $(\text{supp } \sigma_\alpha, |\cdot|, \sigma_\alpha)$  is the doubling metric space, Theorem 2' and Theorem B in [14] imply Theorem 2. To make the paper more self-contained, we give a proof of this implication.

**Proof.** Let  $\theta$  be a one-component inner function. We first remark that the integral in formula (4) is correctly defined for  $F \in K_\theta^\infty$  and  $b \in \text{BMO}(\sigma_\alpha)$ . Indeed, by Lemma 2.1 and Lemma 3.3 every function  $F \in K_\theta^1$  has the trace  $f$  on the set  $a(\sigma_\alpha)$  of full measure  $\sigma_\alpha$ . If  $F \in K_\theta^1 \cap zH^\infty$ , then  $f \in L^\infty(\sigma_\alpha)$ . Since  $\text{BMO}(\sigma_\alpha) \subset L^1(\mathbb{T})$  the integral in formula (4) converges absolutely.

Consider a continuous linear functional  $\Phi$  on  $K_\theta^1 \cap zH^1$ . Since  $K_\theta^2 \subset K_\theta^1$  and  $K_\theta^2 \cap zH^2$  is the Hilbert space there exists a function  $G \in K_\theta^2 \cap zH^2$  such that  $\Phi(F) = \int_{\mathbb{T}} F \bar{G} dm$  for all  $F \in K_\theta^2 \cap zH^2$ . Denote by  $b$  the restriction of the function  $\bar{G}$  to the set  $a(\sigma_\alpha)$  of full measure  $\sigma_\alpha$ . By Clark theorem, we have  $b \in L^2(\sigma_\alpha)$ . Let us prove that  $b \in \text{BMO}(\sigma_\alpha)$ . For every function  $F \in K_\theta^2 \cap zH^2$  we have

$$\Phi(F) = \int_{\mathbb{T}} F \bar{G} dm = \int_{\mathbb{T}} F b d\sigma_\alpha = \Phi_b(F), \quad (34)$$

where we use again Clark theorem. Take an arc  $\Delta$  of  $\mathbb{T}$  and consider the function  $a_0 \in L^\infty(\sigma_\alpha)$  such that  $|a_0| = 1$ ,  $a_0(b - \langle b \rangle_{\Delta, \sigma_\alpha}) = |a_0(b - \langle b \rangle_{\Delta, \sigma_\alpha})|$   $\sigma_\alpha$ -almost

everywhere on  $\Delta$  and  $a = 0$   $\sigma_\alpha$ -everywhere off  $\Delta$ . Denote by  $\chi_\Delta$  the indicator of the set  $\Delta$ . The function

$$a = \frac{1}{2\sigma_\alpha(\Delta)}(a_0 - \langle a_0 \rangle_{\Delta, \sigma_\alpha})\chi_\Delta$$

is an atom with respect to the measure  $\sigma_\alpha$  and we have

$$\int_{\mathbb{T}} ab \, d\sigma_\alpha = \int_{\Delta} a(b - \langle b \rangle_{\Delta, \sigma_\alpha}) \, d\sigma_\alpha = \frac{1}{2\sigma_\alpha(\Delta)} \int_{\Delta} |b - \langle b \rangle_{\Delta, \sigma_\alpha}| \, d\sigma_\alpha. \quad (35)$$

By Theorem 2' the function  $a$  can be continued analytically to  $\mathbb{D}$  as a function  $F_a \in K_\theta^1 \cap zH^1$  with  $\|F_a\|_{L^1(\mathbb{T})} \lesssim 1$ . Since  $a \in L^2(\sigma_\alpha)$ , we have  $F_a \in K_\theta^2 \cap zH^2$  by Clark theorem. Now it follows from (34) and (35) that  $\|b\|_{\sigma_\alpha^*} \lesssim \|\Phi_b\|$ .

Conversely, take a function  $b \in \text{BMO}(\sigma_\alpha)$  and consider the functional  $\Phi_b$  densely defined on  $K_\theta^1 \cap zH^1$  by formula (4). For every  $\sigma_\alpha$ -atom  $a$  supported on an arc  $\Delta$  we have

$$\left| \int_{\mathbb{T}} ab \, d\sigma_\alpha \right| = \left| \int_{\Delta} a(b - \langle b \rangle_{\Delta, \sigma_\alpha}) \, d\sigma_\alpha \right| \leq \frac{1}{\sigma_\alpha(\Delta)} \int_{\Delta} |b - \langle b \rangle_{\Delta, \sigma_\alpha}| \, d\sigma_\alpha. \quad (36)$$

This shows that the functional  $f \mapsto \int_{\mathbb{T}} fb \, d\sigma_\alpha$  is continuous on  $H_{at}^1(\sigma_\alpha)$ . By Theorem 2', the restriction of every function  $F \in K_\theta^1 \cap zH^1$  to  $a(\sigma_\alpha)$  belongs to  $H_{at}^1(\sigma_\alpha)$  and  $\|F\|_{H_{at}^1(\sigma_\alpha)} \lesssim \|F\|_{L^1(\mathbb{T})}$ . Hence the functional  $\Phi_b$  is continuous on  $K_\theta^1 \cap zH^1$  and we see from (36) that  $\|\Phi_b\| \lesssim \|b\|_{\sigma_\alpha^*}$ .  $\square$

#### 4. TRUNCATED HANKEL AND TOEPLITZ OPERATORS

Let  $\theta$  be an inner function. Denote by  $P_\theta$  the orthogonal projection in  $L^2(\mathbb{T})$  to the subspace  $K_\theta^2$ . The truncated Toeplitz operator  $A_\psi : K_\theta^2 \rightarrow K_\theta^2$  with symbol  $\psi \in L^2(\mathbb{T})$  is densely defined by

$$A_\psi : f \mapsto P_\theta(\psi f), \quad f \in K_\theta^\infty.$$

Truncated Toeplitz and Hankel operators are closely related. Indeed, the antilinear isometry  $g \mapsto \bar{z}\theta\bar{g}$  on  $L^2(\mathbb{T})$  preserves the subspace  $K_\theta^2$  and for every  $f, g \in K_\theta^\infty$  we have

$$(A_\psi f, g) = (\psi f, g) = (\Gamma_{\bar{\theta}\psi} f, \overline{zg_1}), \quad g_1 = \bar{z}\theta\bar{g}. \quad (37)$$

This shows that the operators  $A_\psi, \Gamma_{\bar{\theta}\psi}$  are bounded (compact, of trace class, etc.) or not simultaneously and  $\|A_\psi\| = \|\Gamma_{\bar{\theta}\psi}\|$ . Below we briefly discuss some results related to the boundedness problem for truncated Toeplitz operators.

We will say that the truncated Toeplitz operator  $A_\psi$  has a bounded symbol  $\psi_1$  if  $A_\psi = A_{\psi_1}$  for a function  $\psi_1 \in L^\infty(\mathbb{T})$ . It can be shown all symbols of the zero truncated Toeplitz operator on  $K_\theta^2$  have the form  $\bar{\theta}g_1 + \theta g_2$ , where  $g_1, g_2 \in H^2$ , see [25]. Hence the operator  $A_\psi : K_\theta^2 \rightarrow K_\theta^2$  has a bounded symbol if and only if the set  $\psi + \bar{\theta}H^2 + \theta H^2$  contains a bounded function on  $\mathbb{T}$ . Clearly, every truncated Toeplitz operator with bounded symbol is bounded. The following question arises: does every bounded truncated Toeplitz operator have a bounded symbol?



**4.1. Analytic symbols.** In 1967, D. Sarason [24] described the commutant  $\{S_\theta\}'$  of the restricted shift operator  $S_\theta : f \mapsto P_\theta(zf)$  on  $K_\theta^2$ . He proved that a bounded operator  $A$  on  $K_\theta^2$  commutes with  $S_\theta$  if and only if there exists a function  $\psi \in H^\infty$  such that  $A = A_\psi$ . Moreover, we have  $\|A_\psi\| = \text{dist}_{H^\infty}(\psi, \theta H^\infty)$  and one can choose the function  $\psi$  so that  $\|A\| = \|\psi\|_{H^\infty}$ . This well-known theorem yields a boundedness criterium for truncated Toeplitz operators with analytic symbols. Indeed, for every  $\psi \in H^2$  and  $f \in K_\theta^\infty$  we have  $A_\psi S_\theta f = S_\theta A_\psi f$ . Hence the operator  $A_\psi$  is bounded if and only if  $A_\psi \in \{S_\theta\}'$  which is equivalent to the existence of a function  $\psi_1 \in H^\infty$  such that  $A_\psi = A_{\psi_1}$  (in other words, we have  $\psi + \theta h \in H^\infty$  for some  $h \in H^2$ ). The equality  $\|A_\psi\| = \text{dist}_{H^\infty}(\psi, \theta H^\infty)$  for  $\psi \in H^\infty$  leads to a short proof for the Nevanlinna-Pick interpolation theorem and its generalization, see [24].

It was observed by N. K. Nikolskii that many problems for truncated Toeplitz operators with analytic symbols can be easily reduced to the problems for usual Hankel operators on  $H^2$ . The reduction is based on the fact that for every  $\psi \in H^2$  the operator  $\bar{\theta}A_\psi P_\theta$  from  $H^2$  to  $\overline{zH^2}$  coincides with the Hankel operator  $H_{\bar{\theta}\psi}$ . In particular, the operator  $A_\psi$  is bounded (compact, of trace class, etc.) if and only if so is the operator  $H_{\bar{\theta}\psi}$ . Since Hankel operators on  $H^2$  are well studied this observation immediately yields consequences for truncated Toeplitz operators. As an example, the operator  $A_\psi$  on  $K_\theta^2$  with symbol  $\psi \in H^2$  is compact if and only if  $\bar{\theta}\psi \in C(\mathbb{T}) + H^2$ , where  $C(\mathbb{T})$  denotes the algebra of continuous functions on the unit circle  $\mathbb{T}$ . For more information see Lecture 8 in [19] and Section 1.2 in [20].

**4.2. General symbols.** Until recently, a little was known about truncated Toeplitz operators with general symbols in  $L^2(\mathbb{T})$ . For such operators the boundedness problem is more complicated.

In 1987, R. Rochberg [23] proved that every bounded Toeplitz operator on the Paley-Wiener space  $\text{PW}_{[-a,a]}^2$  has a bounded symbol. Using the Fourier transform, he reduced the general case of the problem to consideration of the Toeplitz operators on  $\text{PW}_{[0,a]}^2$  with analytic symbols. Recently, M. Carlsson [9] use a result from [23] to prove the boundedness criterium for Toeplitz and Hankel operators on  $\text{PW}_{[-a,a]}^2$  in terms of  $\text{BMO}(\frac{\pi}{a}\mathbb{Z})$ , see Section 1.

Every finite Toeplitz matrix  $A$  clearly have bounded symbols. However, the question concerning the best possible constant  $c_A$  in the inequality

$$\inf\{\|\psi\|_{L^\infty(\mathbb{T})} : A_\psi = A\} \leq c_A \cdot \|A\|$$

is nontrivial. In 2001, M. Bakonyi and D. Timotin proved that  $c_A \leq 2$  for every self-adjoint finite Toeplitz matrix  $A$ . As a corollary, we have  $c_A \leq 4$  for a general finite Toeplitz matrix  $A$  that was improved to  $c_A \leq 3$  by L. N. Nikolskaya and Yu. B. Farforovskaya [18] in 2003. Next, in 2007 D. Sarason [25] compute  $c_A = \pi/2$  for  $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$  and proved that  $c_A \leq \pi/2$  for every  $2 \times 2$  self-adjoint Toeplitz matrix  $A$ . In paper [27] A. L. Volberg discuss several approaches to the dual version of the problem of determining  $\sup_A c_A$  over all finite Toeplitz matrices  $A$ , which can be formulated in terms of weak factorizations of analytic polynomials.

In 2010, A. D. Baranov, I. Chalendar, E. Fricain, J. Mashregi, and D. Timotin [6] constructed an inner function  $\theta$  and a bounded truncated Toeplitz operator  $A$  on  $K_\theta^2$  that has no bounded symbols. Shortly after that in [5] appeared a

description of coinvariant subspaces  $K_\theta^2$  on which every bounded truncated Toeplitz operator has a bounded symbol. The proof in [5] is based on a duality relation between the space of all bounded truncated Toeplitz operators on  $K_\theta^2$  and a special function space. With help of formula (37) it is easy to reformulate the results of [5] for truncated Hankel operators. We do this below as a preparation to the proof of Theorem 3.

**4.3. Duality for truncated Hankel operators.** Let  $\theta$  be an inner function. Consider the linear space

$$Y_\theta = \left\{ \sum_{k=0}^{\infty} x_k y_k, \ x_k \in K_\theta^2, \ y_k \in zK_\theta^2, \ \sum_{k=0}^{\infty} \|x_k\|_{L^2(\mathbb{T})} \|y_k\|_{L^2(\mathbb{T})} < \infty \right\}.$$

As is easy to see, we have  $Y_\theta \subset K_{\theta^2}^1 \cap zH^1$ . Define the norm in  $Y_\theta$  by

$$\|h\|_{Y_\theta} = \inf \left\{ \sum_{k=0}^{\infty} \|x_k\|_{L^2(\mathbb{T})} \|y_k\|_{L^2(\mathbb{T})} : h = \sum_{k=0}^{\infty} x_k y_k, \ x_k \in K_\theta^2, \ y_k \in zK_\theta^2 \right\}.$$

With this norm  $Y_\theta$  is a Banach space. Denote by  $\mathcal{H}_\theta$  the linear space of all bounded truncated Hankel operators acting from  $K_\theta^2$  to  $\overline{zK_\theta^2}$ . It follows from Theorem 4.2 in [25] that  $\mathcal{H}_\theta$  is closed in the weak operator topology. Hence  $\mathcal{H}_\theta$  is the Banach space under the standard operator norm and moreover it has a predual space. It follows from Theorem 2.3 of [5] that  $Y_\theta^* = \mathcal{H}_\theta$ . That is, for every continuous linear functional  $\Psi$  on  $Y_\theta$  there exists the unique operator  $\Gamma \in \mathcal{H}_\theta$  such that  $\Psi = \Psi_\Gamma$ , where

$$\Psi_\Gamma : h \mapsto \sum_{k=0}^{\infty} (\Gamma x_k, \overline{y_k}), \quad h \in Y_\theta, \quad h = \sum_{k=0}^{\infty} x_k y_k. \quad (38)$$

Conversely, for every operator  $\Gamma \in \mathcal{H}_\theta$  the mapping  $\Psi_\Gamma$  is the correctly defined continuous linear functional on the space  $Y_\theta$  and we have  $\|\Psi_\Gamma\| = \|\Gamma\|$ .

With help of the equality  $Y_\theta^* = \mathcal{H}_\theta$  the boundedness problem for truncated Hankel operators can be reformulated in terms of function theory. Indeed, now it is easy to see from Hahn-Banach theorem that every bounded truncated Hankel operator on  $K_\theta^2$  has a bounded symbol if and only if  $Y_\theta$  is a closed subspace of  $L^1(\mathbb{T})$ , in which case  $Y_\theta$  coincides with  $K_{\theta^2}^1 \cap zH^1$  as a set, see details in [5]. Note that if  $Y_\theta = K_{\theta^2}^1 \cap zH^1$  as set, then the norms  $\|\cdot\|_{Y_\theta}$  and  $\|\cdot\|_{L^1(\mathbb{T})}$  are equivalent on  $Y_\theta$ . It was proved in [5] that  $Y_\theta = K_{\theta^2}^1 \cap zH^1$  for every one-component inner function  $\theta$ .

Thus, we see from the results of [5] and Theorem 2 that for every one-component inner function  $\theta$  we have

$$Y_\theta^* = \mathcal{H}_\theta, \quad Y_\theta = K_{\theta^2}^1 \cap zH^1, \quad (K_{\theta^2}^1 \cap zH^1)^* = \text{BMO}(\nu_\alpha),$$

where  $\nu_\alpha$  is the Clark measure of the inner function  $\theta^2$ . It remains to combine this relations to obtain Theorem 3.

**4.4. Proof of Theorem 3.** Let  $\theta$  be a one-component inner function and let  $\Gamma_\varphi$  be a truncated Hankel operator on  $K_\theta^2$  with standard symbol  $\varphi \in \overline{K_{\theta^2}^2 \cap zH^2}$ ; we do not

assume now that the operator  $\Gamma_\varphi$  is bounded. For every function  $h = \sum_{k=0}^{\infty} x_k y_k$  in  $Y_\theta \cap L^\infty(\mathbb{T})$  we have

$$\Psi_{\Gamma_\varphi}(h) = \sum_{k=0}^{\infty} (\Gamma_\varphi x_k, \overline{y_k}) = \int_{\mathbb{T}} \varphi \sum_{k=0}^{\infty} x_k y_k dm = \int_{\mathbb{T}} h \varphi dm = \int_{\mathbb{T}} h \varphi d\nu_\alpha, \quad (39)$$

where the last equality follows from Clark theorem for the inner function  $\theta^2$ . We see that  $\Psi_{\Gamma_\varphi}$  coincides on  $Y_\theta \cap L^\infty(\mathbb{T})$  with the functional

$$\Phi_\varphi : h \mapsto \int_{\mathbb{T}} h \varphi d\nu_\alpha, \quad h \in K_{\theta^2}^1 \cap zH^\infty.$$

Since the inner function  $\theta^2$  is one-component the Banach spaces  $Y_\theta$  and  $K_{\theta^2}^1 \cap zH^1$  coincide as sets and their norms are equivalent. It follows that the densely defined functionals  $\Psi_{\Gamma_\varphi} : Y_\theta \rightarrow \mathbb{C}$  and  $\Phi_\varphi : K_{\theta^2}^1 \cap zH^1 \rightarrow \mathbb{C}$  are bounded or not simultaneously and  $\|\Psi_{\Gamma_\varphi}\| \asymp \|\Phi_\varphi\|$ , where the constants involved depend only on  $\theta$ . By Theorem 2 for the inner function  $\theta^2$  the functional  $\Phi_\varphi$  is bounded if and only if  $\varphi \in \text{BMO}(\nu_\alpha)$ , and in the latter case we have  $\|\Phi_\varphi\| \asymp \|\varphi\|_{\nu_\alpha^*}$ . Now result follows from the equality  $\|\Gamma_\varphi\| = \|\Psi_{\Gamma_\varphi}\|$ .  $\square$

**4.5. Compact truncated Hankel operators.** Let  $\mu$  be a measure on  $\mathbb{T}$  with properties (a) – (c). For every  $b \in \text{BMO}(\mu)$  define

$$M_\varepsilon(b) = \sup \left\{ \frac{1}{\mu(\Delta)} \int_{\Delta} |b - \langle b \rangle_{\Delta, \mu}| d\mu, \Delta \text{ is an arc of } \mathbb{T} \text{ with } 0 < \mu(\Delta) \leq \varepsilon \right\}.$$

Consider the space  $\text{VMO}(\mu) = \{b \in \text{BMO}(\mu) : \lim_{\varepsilon \rightarrow 0} M_\varepsilon(b) = 0\}$  of functions of vanishing mean oscillation with respect to the measure  $\mu$ . It can be shown that  $\text{VMO}(\mu)$  is the closure in  $\text{BMO}(\mu)$  of the set of all finitely supported sequences.

**Proposition 4.1.** *Let  $\theta$  be a one-component inner function, and let  $\nu_\alpha$  be the Clark measure of the inner function  $\theta^2$ . The truncated Hankel operator  $\Gamma_\varphi : K_\theta^2 \rightarrow \overline{zK_\theta^2}$  with standard symbol  $\varphi$  is compact if and only if  $\varphi \in \text{VMO}(\nu_\alpha)$ .*

**Proof.** It follows from Theorem 2.3 of [5] that  $(\mathcal{H}_\theta \cap S_\infty)^* = Y_\theta$ , where  $S_\infty$  denotes the ideal of all compact operators acting from  $K_\theta^2$  to  $\overline{zK_\theta^2}$ . Hence a bounded truncated Hankel operator  $\Gamma$  on  $K_\theta^2$  is compact if and only if the functional  $\Psi_\Gamma$  in (38) is continuous in the weak\* topology on  $Y_\theta$ . Let  $\varphi$  be the standard symbol of the operator  $\Gamma_\varphi$ . By Corollary 2.5 in [5] and Theorem 2' we have

$$Y_\theta = K_{\theta^2}^1 \cap zH^1, \quad V_\alpha(K_{\theta^2}^1 \cap zH^1) = H_{at}^1(\nu_\alpha).$$

From formula (39) we see that the operator  $\Gamma_\varphi$  is compact if and only if the restriction of  $\varphi$  to  $a(\nu_\alpha)$  generates the weak\* continuous functional  $\Phi_\varphi : f \mapsto \int f \varphi d\nu_\alpha$  on the space  $H_{at}^1(\nu_\alpha)$ . For any doubling measure  $\mu$  we have  $\text{VMO}(\mu)^* = H_{at}^1(\mu)$ , see Theorem 4.1 in [14]. It follows that  $\Gamma_\varphi \in S_\infty$  if and only if  $\varphi \in \text{VMO}(\nu_\alpha)$ .  $\square$

**4.6. Functions in  $K_\theta^2$  of bounded mean oscillation.** Theorem 3 provides the following description of functions in  $K_\theta^2 \cap \text{BMO}(\mathbb{T})$ .

**Proposition 4.2.** *Let  $\theta$  be a one-component inner function and let  $\varphi \in K_\theta^2$ . Then we have  $\varphi \in K_\theta^2 \cap \text{BMO}(\mathbb{T})$  if and only if  $\varphi \in \text{BMO}(\nu_\alpha)$ , where  $\nu_\alpha$  is the Clark measure of the inner function  $\theta^2$ .*

**Proof.** A function  $\varphi \in H^2$  belongs to the space  $\text{BMO}(\mathbb{T})$  if and only if the Hankel operator  $H_{\bar{\varphi}} : H^2 \rightarrow \overline{zH^2}$  is bounded, see Theorem 1.2 in Chapter 1 of [20]. Assume that  $\varphi \in K_{\theta}^2$  and consider the truncated Hankel operator  $\Gamma_{\varphi} : K_{\theta}^2 \rightarrow \overline{zK_{\theta}^2}$ . For every function  $f \in K_{\theta}^{\infty}$  we have  $\bar{\varphi}f \in \bar{\theta}H^2$ . Hence,

$$H_{\bar{\varphi}}f = P_{-}(\bar{\varphi}f) = P_{\bar{\theta}}(\bar{\varphi}f) = \Gamma_{\bar{\varphi}}f, \quad f \in K_{\theta}^{\infty}.$$

Also,  $H_{\varphi}f = 0$  for all  $f \in \theta H^{\infty}$ . Therefore the operators  $H_{\bar{\varphi}}$  and  $\Gamma_{\bar{\varphi}}$  are bounded or not simultaneously and  $\|H_{\bar{\varphi}}\| = \|\Gamma_{\bar{\varphi}}\|$ . Now the result follows from Theorem 3.  $\square$

**4.7. Finite Hankel and Toeplitz matrices.** Let  $\Gamma = (\gamma_{j+k})_{0 \leq j, k \leq n-1}$  be a Hankel matrix of size  $n \times n$ . Associate with  $\Gamma$  the antianalytic polynomial

$$\varphi = \gamma_0 \bar{z} + \gamma_1 \bar{z}^2 + \dots \gamma_{2n-2} \bar{z}^{2n-1}.$$

For the inner function  $\theta_n = z^n$  the space  $K_{\theta_n}^2$  consists of analytic polynomials of degree at most  $n-1$ . Consider the truncated Hankel operator  $\Gamma_{\varphi} : K_{\theta_n}^2 \rightarrow \overline{zK_{\theta_n}^2}$ ,

$$\Gamma_{\varphi} : f \mapsto P_{\theta_n}(\varphi f), \quad f \in K_{\theta_n}^2.$$

We have  $(\Gamma_{\varphi} z^j, \bar{z}^{k+1}) = \gamma_{j+k}$  for every  $0 \leq j, k \leq n-1$ . It follows that the matrix  $\Gamma$  as the operator on  $\mathbb{C}^n$  is unitary equivalent to the operator  $\Gamma_{\varphi}$ . Analogously, the Toeplitz matrix  $A = (\alpha_{j-k})_{0 \leq j, k \leq n-1}$  is unitarily equivalent to the truncated Toeplitz operator  $A_{\psi} : K_{\theta_n}^2 \rightarrow K_{\theta_n}^2$  with symbol

$$\psi = \alpha_{-(n-1)} \bar{z}^{n-1} + \dots + \alpha_{n-1} z^{n-1}.$$

If moreover  $\alpha_m = \gamma_{(n-1)-m}$  for all  $m \in \mathbb{Z}$  with  $|m| \leq n-1$ , then  $\varphi = \bar{\theta}_n \psi$  and we have  $\|\Gamma\| = \|A\|$  by formula (37). Consider the measure

$$\mu_{2n} = \frac{1}{2n} \sum \delta_{\sqrt[n]{1}}$$

equally distributed at the roots of identity of order  $2n$ :  $\text{supp } \mu = \{\xi \in \mathbb{T} : \xi^{2n} = 1\}$ . Let  $c_{1,n}, c_{2,n}$  be the best possible constants in the inequality

$$c_{1,n} \|\varphi\|_{\mu_{2n}^*} \leq \|\Gamma_{\varphi}\| \leq c_{2,n} \|\varphi\|_{\mu_{2n}^*}, \quad (40)$$

where  $\Gamma_{\varphi}$  runs over all truncated Hankel operators on  $K_{\theta_n}^2$ ,  $\varphi$  is the standard symbol of  $\Gamma_{\varphi}$ . Corollary 1 of Theorem 3 claims that the sequences  $\{c_{1,n}\}_{n \geq 1}$  and  $\{c_{2,n}\}_{n \geq 1}$  are bounded. We prove this below.

**Proof of Corollary 1.** We may assume that  $n \geq 2$ . It follows from Lemma 2.1 that  $\mu_{2n}$  is the Clark measure  $\nu_1$  of the inner function  $\theta_n^2 = z^{2n}$ . This allows us to estimate the constants in formula (40) using the proofs of Theorem 2 and Theorem 3. Denote

$$\begin{aligned} d'_n &= \sup\{\|h\|_{L^1(\mathbb{T})}, h \in K_{\theta_n^2}^1 \cap zH^1, \|h\|_{H_{at}^1(\mu_{2n})} \leq 1\}; \\ d''_n &= \sup\{\|h\|_{Y_{\theta_n}}, h \in K_{\theta_n^2}^1 \cap zH^1, \|h\|_{L^1(\mathbb{T})} \leq 1\}. \end{aligned} \quad (41)$$

Let  $\Gamma_{\varphi} : K_{\theta_n}^2 \rightarrow \overline{zK_{\theta_n}^2}$  be a truncated Hankel operator with standard symbol  $\varphi$ . Consider the functional  $\Psi : h \mapsto \int_{\mathbb{T}} h \varphi d\mu_{2n}$  on the Banach space  $Y_{\theta_n}$ . From formula (35) and the equality  $\|\Psi\| = \|\Gamma_{\varphi}\|$  (see Section 4.3) we obtain

$$\begin{aligned} \|\varphi\|_{\mu_{2n}^*} &\leq 2 \sup\{|\Psi(h)|, \|h\|_{H_{at}^1(\mu_{2n})} \leq 1\} \leq 2d'_n \sup\{|\Psi(h)|, \|h\|_{L^1(\mathbb{T})} \leq 1\} \\ &\leq 2d'_n d''_n \sup\{|\Psi(h)|, \|h\|_{Y_{\theta_n}} \leq 1\} = 2d'_n d''_n \|\Gamma_{\varphi}\|. \end{aligned} \quad (42)$$

Hence,  $c_{1,n}^{-1} \leq 2d'_n \cdot d''_n$ . It follows from the results of Nikolskaya and Farforovskaya [18] that  $d''_n \leq 3$ , see also Section 1.2 in [27]. To estimate the constant  $d'_n$  assume that the restriction of  $f \in K_{\theta_n}^1 \cap zH^1$  to  $a(\mu_{2n})$  is a  $\mu_{2n}$ -atom supported on a closed arc  $\Delta$  of the unit circle  $\mathbb{T}$  with center  $\xi_c$  and endpoints in  $a(\mu_{2n})$ . Let  $\{\nu_\beta^n\}_{|\beta|=1}$  be the family of the Clark measures of the inner function  $\theta_n^2$ ; we have  $\nu_1^n = \mu_{2n}$ . Combining formulas (24) and (25) in the proof of Theorem 2, we obtain

$$\|f\|_{L^1(\mathbb{T})} \leq \sup_{|\beta|=1} \left( \sqrt{\frac{\nu_\beta^n(2\Delta)}{\nu_1^n(\Delta)}} + 4\pi m(\Delta) \int_{\mathbb{T} \setminus 2\Delta} \frac{1}{|z - \xi_c|^2} d\nu_\beta^n(z) \right).$$

Observe that  $\nu_1^n(\Delta) \geq m(\Delta)$  and  $\nu_\beta^n(2\Delta) \leq 3m(\Delta)$ . Let  $\xi_1, \xi_2$  be the nearest points to  $\xi_c$  in  $a(\nu_\beta^n) \setminus 2\Delta$ . Then  $|\xi_c - \xi_{1,2}| \geq \text{diam}(2\Delta)/2 \geq m(2\Delta)$  and we have

$$\begin{aligned} \int_{\mathbb{T} \setminus 2\Delta} \frac{d\nu_\beta^n(z)}{|z - \xi_c|^2} &\leq \int_{\mathbb{T} \setminus 2\Delta} \frac{dm(z)}{|z - \xi_c|^2} + \frac{1}{2n} \left( \frac{1}{|\xi_c - \xi_1|^2} + \frac{1}{|\xi_c - \xi_2|^2} \right) \\ &\leq \frac{\pi}{4m(2\Delta)} + \frac{1}{2m(2\Delta)} < \frac{1}{m(\Delta)}. \end{aligned}$$

Hence,  $\|f\|_{L^1(\mathbb{T})} \leq \sqrt{3} + 4\pi < 15$ . This gives us  $d'_n < 15$  and  $c_{1,n}^{-1} < 90$ .

Let us turn to the second inequality in (40). As before, from formula (36) we obtain

$$\begin{aligned} \|\Gamma_\varphi\| &= \sup \{ |\Psi(h)|, \|h\|_{Y_{\theta_n}} \leq 1 \} \leq D'_n \sup \{ |\Psi(h)|, \|h\|_{L^1(\mathbb{T})} \leq 1 \} \\ &\leq D'_n D''_n \sup \{ |\Psi(h)|, \|h\|_{H_{at}^1(\mu_{2n})} \leq 1 \} \leq D'_n D''_n \|\varphi\|_{\mu_{2n}^*}, \end{aligned} \quad (43)$$

where

$$\begin{aligned} D'_n &= \sup \{ \|h\|_{H_{at}^1(\mu_{2n})}, h \in K_{\theta_n}^1 \cap zH^1, \|h\|_{L^1(\mathbb{T})} \leq 1 \}; \\ D''_n &= \sup \{ \|h\|_{L^1(\mathbb{T})}, h \in K_{\theta_n}^1 \cap zH^1, \|h\|_{Y_{\theta_n}} \leq 1 \}. \end{aligned} \quad (44)$$

By the Cauchy-Schwarz inequality,  $D''_n \leq 1$ . In the proof of Theorem 2' we have seen that  $D'_n \leq 24c_{5n}c_{6n}M$ , where  $M$  is the norm of the non-tangential maximal operator  $F \mapsto F^*$  on  $H^1$  and  $c_{5n}, c_{6n}$  are the constants  $c_5, c_6$  from Lemma 3.5 for the inner function  $\theta = \theta_n$ . Since  $\tilde{A}_{\mu_{2n}} = \tilde{B}_{\mu_{2n}} = 1$ , we have  $D'_n \leq 24 \cdot 4 \cdot 60 \cdot M \cdot \varepsilon_n^{-1}$ , where  $\varepsilon_n$  stands for the parameter  $\varepsilon$  in Lemma 3.2 for  $\theta = \theta_n$ . Next, since the sublevel set  $\Omega_\delta$  of  $\theta_n$  is connected for every  $\delta > 0$ , the proof of Lemma 3.2 shows that one can take  $\varepsilon_n = \kappa_n/2$ , where  $\kappa_n \leq \kappa_n^* = (2\tilde{B}_{\mu_{2n}})^{-1} = 1/2$  is chosen so that estimate (14) holds for  $\theta = \theta_n^2$ ,  $\kappa = \kappa_n$ . It remains to show that  $\inf_n \kappa_n > 0$ . For this aim it is sufficient to prove that the functions  $f_{\xi_0, n} = f_{\xi_0}$  in formula (17) for  $\theta = \theta_n$  are bounded uniformly in  $n$ . By formula (13),  $C_{\mu_{2n}} \leq 1/2$ . Next, for every pair of atoms  $\xi, \xi_0 \in a(\mu_{2n})$  and for all  $z \in D_{\xi_0}(\kappa_n^*)$  we have  $|\xi - z| \geq |\xi - \xi_0|/2$ . Since  $\inf_n A_{\mu_{2n}} = A_{\mu_4} = \frac{1}{4\sqrt{2}}$  and  $B_{\mu_{2n}} \leq \tilde{B}_{\mu_{2n}} = 1$  we see that estimate (16) for  $\sigma_\alpha = \mu_{2n}$  takes the following form:

$$\begin{aligned} \int_{\mathbb{T} \setminus \{\xi_0\}} \frac{d\mu_{2n}(\xi)}{|\xi - \xi_0|^2} &\leq 2\pi \int_{\mathbb{T} \setminus \Delta} \frac{d\mu_{2n}(\xi)}{|\xi - \xi_0|^2} + \frac{64}{\mu_{2n}\{\xi_0\}} \\ &\leq \frac{\pi^2}{2m(\Delta)} + \frac{64}{\mu_{2n}\{\xi_0\}} \\ &\leq \left( 64 + \frac{\pi^2}{4} \right) \frac{1}{\mu_{2n}\{\xi_0\}}. \end{aligned}$$

It follows that  $|f_{\xi_0}| < 70$  on  $D_{\xi_0}$  and estimate (14) holds for  $\theta = \theta_n$  with any constant  $\kappa_n \leq \kappa_n^*$  such that  $70 \leq (2\kappa_n)^{-1}$ . In particular, one can take  $\kappa_n = 1/140$  for all  $n \geq 2$ . We now see that the constants  $c_{2,n}$  are bounded:  $c_{2,n} \leq D'_n \leq 24 \cdot 4 \cdot 60 \cdot 280 \cdot M < 10^7 M$ .  $\square$

**Corollary 2.** *Let  $A = (\alpha_{j-k})_{0 \leq k, j \leq n-1}$  be a Toeplitz matrix of size  $n \times n$ ; consider its standard symbol  $\psi = \alpha_{-(n-1)} \bar{z}^{n-1} + \dots + \alpha_{n-1} z^{n-1}$ . We have*

$$c_1 \|\bar{z}^n \psi\|_{\mu_{2n}^*} \leq \|A\| \leq c_2 \|\bar{z}^n \psi\|_{\mu_{2n}^*},$$

where the constants  $c_1, c_2$  do not depend on  $n$ .

The author failed to find a simple argument allowing obtain Corollary 1 from the BMO-criterium for the boundedness of Hankel operators on  $H^2$ . The inverse implication is quite elementary.

**Proposition 4.3.** *Let  $\varphi \in \overline{zH^2}$ . The Hankel operator  $H_\varphi : H^2 \rightarrow \overline{zH^2}$  is bounded if and only if  $\varphi \in \text{BMO}(\mathbb{T})$ . Moreover we have  $c_1 \|\varphi\|_* \leq \|H_\varphi\| \leq c_2 \|\varphi\|_*$  with constants  $c_1, c_2$  from Corollary 1.*

**Proof.** Let  $H_\varphi : H^2 \rightarrow \overline{zH^2}$  be a bounded Hankel operator on  $H^2$  with symbol  $\varphi \in \overline{zH^2}$ . Then there are finite-rank Hankel operators  $H_{\varphi_n}$ ,  $\varphi_n \in \overline{K_{\theta_n}^2} \cap \overline{zH^2}$ , such that  $H_\varphi$  is the limit of  $H_{\varphi_n}$  in the weak\* operator topology. Moreover one can choose  $H_{\varphi_n}$  so that  $\sup_n \|H_{\varphi_n}\| \leq \|H_\varphi\|$ . For every  $n \geq 1$  and  $k \geq n$  the operator norm of the Hankel operator  $H_{\varphi_n}$  is equal to the operator norm of the truncated Hankel operator on  $K_{\theta_k}^2$  with symbol  $\varphi_n$ , where  $\theta_k = z^k$ . Since  $\|\varphi_n\|_* = \lim_{k \rightarrow \infty} \|\varphi_n\|_{\mu_{2k}^*}$  we see from Corollary 1 that

$$c_1 \|\varphi_n\|_* \leq \|H_{\varphi_n}\| \leq c_2 \|\varphi_n\|_*, \quad n \geq 1. \quad (45)$$

It follows that  $c_1 \sup \|\varphi_n\|_* \leq \|H_\varphi\|$ . Since  $H_{\varphi_n}$  tend to  $H_\varphi$  in the weak\* operator topology we have  $\lim_{n \rightarrow \infty} \int_{\mathbb{T}} p \varphi_n dm = \int_{\mathbb{T}} p \varphi dm$  for every trigonometric polynomial  $p$ . It is well-known that  $H_{at}^1(\mathbb{T})^* = \text{BMO}(\mathbb{T})$  (it worth be mentioned that this fact is much more easier than the Fefferman theorem on  $\text{Re}(zH^1)^* = \text{BMO}(\mathbb{T})$  which is generally used in the proof of the boundedness criterium for Hankel operators). Since trigonometric polynomials are dense in  $\text{BMO}(\mathbb{T})$  in the weak\* topology generated by  $H_{at}^1(\mathbb{T})$ , we have  $\varphi \in \text{BMO}(\mathbb{T})$  and  $c_1 \|\varphi\|_* \leq \|H_\varphi\|$ . Now let  $\varphi \in \overline{zH^2} \cap \text{BMO}(\mathbb{T})$ . Then there are functions  $\varphi_n \in \overline{K_{\theta_n}^2} \cap \overline{zH^2}$  which tend to  $\varphi$  in the weak\* topology of  $\text{BMO}(\mathbb{T})$  and such that  $\sup_n \|\varphi_n\|_* \leq \|\varphi\|_*$ . From (45) we see that  $\|H_{\varphi_n}\| \leq c_2 \|\varphi\|_*$  for the corresponding Hankel operators  $H_{\varphi_n}$ . Since  $L^2(\mathbb{T}) \subset H_{at}^1(\mathbb{T})$  the functions  $\varphi_n$  converge to  $\varphi$  weakly in  $L^2(\mathbb{T})$ . Hence for every pair of analytic polynomials  $p_1, p_2$  we have  $\lim_{n \rightarrow \infty} (H_{\varphi_n} p_1, \overline{z p_2}) = (H_\varphi p_1, \overline{z p_2})$ . It follows that the operators  $H_{\varphi_n}$  converge to the operator  $H_\varphi$  in the weak operator topology and we have  $\|H_\varphi\| \leq c_2 \|\varphi\|_*$ .  $\square$

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